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THERMAL ENTRANCE REGION HEAT TRANSFER FOR RECTANGULAR DUCTS OF VARIOUS ASPECT RATIOS AND PECLET NUMBERS

J. R. DeWitt and W. T. Snyder
ARO, Inc.

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FOREWORD

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This technical report has been reviewed and is approved.

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ABSTRACT

In this investigation the thermal entrance region problem is solved for flow in rectangular ducts of various aspect ratios and for various Peclet numbers. The assumptions under which the problem is solved are steady, fully developed laminar velocity profile, constant fluid properties of viscosity, density, specific heat and thermal conductivity, constant wall temperature, and a uniform inlet fluid temperature. Included in the solution are axial conduction and viscous dissipation. The method of B. G. Galerkin is used to formulate an approximate series solution of the problem. The data presented include bulk mean temperature, local Nusselt number, and the ratio of the local heat-transfer rate to the long wall to the local heat-transfer rate to the short wall. It is concluded that for low Peclet numbers, neglecting the axial conduction term leads to considerable error in the solution.

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NOMENCLATURE

B_n	Eigenvalues given in Eq. (14)
C_p	Specific heat at constant pressure
h	Convective heat-transfer coefficient
K	Thermal conductivity
L_1	Width of duct
L_2	Height of duct
Nu	Nusselt number defined by Eq. (IV-2)
P	Pressure
Pe.	Peclet number defined by Eq. (7)
Q_α	Heat-transfer rate per unit length to α wall
Q_β	Heat-transfer rate per unit length to β wall

R	Aspect ratio defined by Eq. (7)
T	Temperature
T^*	Temperature defined by Eq. (3)
T_f	Viscous dissipation temperatue
T_{ref}	Temperature nondimensionalization factor given by Eq. (7)
T_w	Wall temperature
u	Nondimensionalized fluid velocity
u_m	Nondimensionalized mean velocity defined by Eq. (I-17)
W	Fluid velocity
W_{ref}	Velocity nondimensionalization factor defined by Eq. (7)
X	Space coordinate shown in Fig. 1
Y	Space coordinate shown in Fig. 1
Z	Space coordinate shown in Fig. 1
α	Nondimensionalized space coordinate defined by Eq. (7)
β	Nondimensionalized space coordinate defined by Eq. (7)
θ	Nondimensionalized temperature
θ_f	Nondimensionalized viscous dissipation temperature
θ_m	Nondimensionalized bulk mean temperature
$\Delta\theta_T$	Nondimensionalized temperature difference between inlet and duct wall
θ_w	Nondimensionalized wall temperature
μ	Dynamic viscosity
ξ	Nondimensionalized space coordinate defined by Eq. (7)
ρ	Density
ϕ_i	B. G. Galerkin function set defined by Eqs. (26) and (27)

SUBSCRIPTS

o	Inlet conditions
η	Viscous dissipation component
λ	Nonviscous dissipation component
∞	Fully developed conditions

SECTION I
INTRODUCTION

1.1 BACKGROUND INFORMATION

The problem of the thermal entrance region heat transfer for flat ducts (parallel plates) and circular tubes has received considerable attention in the literature. However, the similar problem for ducts of finite aspect ratio has been investigated far less thoroughly.

The heat transfer in laminar flow of an incompressible fluid with constant properties in the entrance region of a circular tube was first investigated analytically by Graetz (Ref. 1) under the assumption of specified uniform wall temperature, no axial conduction, and no viscous dissipation. Prins, Mulder, and Shenk (Ref. 2) applied the Graetz method to parallel plates under the same assumptions. Various authors (Refs. 3 through 6) have extended the problem to include more complex boundary conditions.

Sparrow (Ref. 7) and Siegel and Sparrow (Ref. 8) solved the parallel plate entrance problem for simultaneous development of velocity and temperature profiles by use of the von Kármán-Pohlhausen method for a range of Prandtl numbers for uniform wall temperature and uniform heat flux, respectively.

Schneider (Ref. 9) included axial conduction in the parallel plate solution for slug flow of various Peclet numbers with finite wall resistance and both uniform and step discontinuity ambient temperature.

Hwang and Fan (Ref. 10) used finite difference techniques to solve the problem of simultaneous development of velocity and temperature profiles for parallel plates for uniform wall temperature and uniform heat flux. Yau and Tien (Ref. 11) solved the same problem for flow of a non-Newtonian fluid with the use of numerical techniques.

Mercer, Pearce, and Hitchcock (Ref. 12) presented experimental data for laminar flow of air between heated plates for Reynolds numbers of 300 to 1500.

For the rectangular duct, Clark and Kays (Ref. 13) used a numerical relaxation method to obtain limiting Nusselt numbers for laminar flow in ducts with aspect ratios of 1, 2, and 5 for constant heat flux and aspect ratios of 1 and 2 for constant wall temperature but without consideration of axial conduction. Sparrow and Siegel (Ref. 14) developed

a variational method for application to rectangular ducts and calculated the first two eigenvalues and eigenfunction constants for the square duct case for constant heat flux but presented no further information.

Dennis, Mercer, and Poots (Ref. 15) used a method similar to the Galerkin method to solve the constant temperature wall case for aspect ratios of 1, 1.5, 2, 4, and 8 but did not include axial conduction.

In this investigation the method of B. G. Galerkin as developed by Eraslan (Ref. 16) for entrance regions is used to formulate approximate solutions for the thermal entrance region of rectangular ducts of various aspect ratios and Peclet numbers. To be included are axial conduction and viscous dissipation.

1.2 STATEMENT OF THE PROBLEM

The purpose of this investigation is to solve the thermal entrance region problem for flow in rectangular ducts of various aspect ratios and for various Peclet numbers. The assumptions under which the problem is to be solved are steady, fully developed laminar velocity profile; constant fluid properties of viscosity, density, specific heat and thermal conductivity; constant wall temperatures; and a uniform inlet fluid temperature. To be included in the solution are viscous dissipation and axial conduction.

The assumption of fully developed laminar flow in a thermal entrance region is applicable for fully developed flow in a duct where the fluid encounters a step change in duct wall temperature. It may also be applied as an approximation for a combination hydrodynamic, thermal entrance region where the velocity profile develops much more rapidly than the thermal profile.

SECTION II
ANALYTIC PROCEDURE

2.1 FORMULATION OF THE PROBLEM WITH GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

The mathematical problem is formulated in the Cartesian coordinate system (X, Y, and Z) with Z taken in the direction of the applied pressure gradient, dP/dZ. The origin of the system is taken at one corner of the rectangular duct and in the plane at which the thermal entrance region is initiated, as shown in Fig. 1.

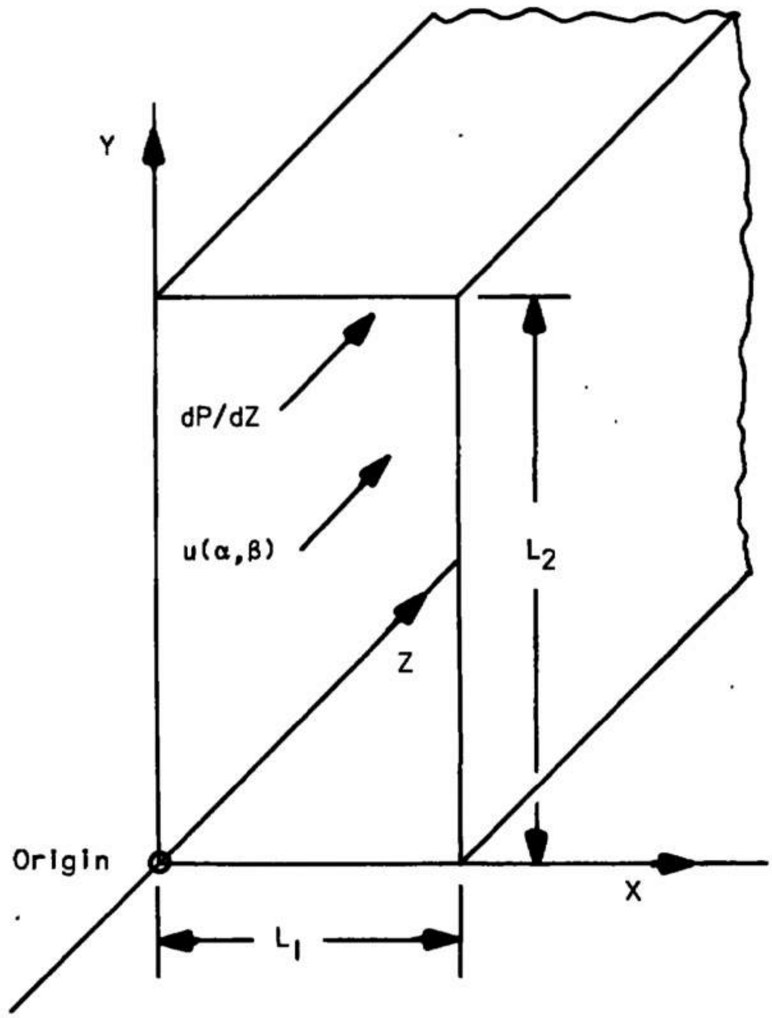

Fig. 1 Flow Geometry

For laminar incompressible flow with constant fluid properties and fully developed velocity profile, the equation of motion may be written as

$$\mu \left(\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} \right) = \frac{dP}{dZ} \tag{1}$$

with the boundary conditions

$$W(\Gamma) = 0$$

where Γ is the boundary of the duct. The energy equation may be written as

$$\rho \; C_p \; W \frac{\partial T}{\partial Z} = K \left(\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} + \frac{\partial^2 T}{\partial Z^2} \right) + \mu \left[\left(\frac{\partial W}{\partial X} \right)^2 + \left(\frac{\partial W}{\partial Y} \right)^2 \right] \tag{2}$$

with boundary conditions

$$T(\Gamma,Z) = T_W$$

$$T(X,Y,0) = T_o(X,Y)$$

Define a new temperature T^* as

$$\begin{aligned} T^* &= T - T_f \\ &= T(X,Y,Z) - T_f(X,Y,Z) \end{aligned} \tag{3}$$

where T_f is the fully developed viscous temperature profile. The energy equation then becomes

$$\begin{aligned} \rho \; C_p \; W \left[\frac{\partial T^*}{\partial Z} + \frac{\partial T_f}{\partial Z} \right] &= K \left[\frac{\partial^2 T^*}{\partial X^2} + \frac{\partial^2 T^*}{\partial Y^2} + \frac{\partial^2 T^*}{\partial Z^2} \right] \\ &+ K \left[\frac{\partial^2 T_f}{\partial X^2} + \frac{\partial^2 T_f}{\partial Y^2} + \frac{\partial^2 T_f}{\partial Z^2} \right] + \left[\mu \left(\frac{\partial W}{\partial X} \right)^2 + \left(\frac{\partial W}{\partial Y} \right)^2 \right] \end{aligned} \tag{4}$$

where T* satisfies the equation

$$\rho\, C_p\, W \frac{\partial T^*}{\partial Z} = K\left[\frac{\partial^2 T^*}{\partial X^2} + \frac{\partial^2 T^*}{\partial Y^2} + \frac{\partial^2 T^*}{\partial Z^2}\right] \tag{5}$$

For constant wall temperature boundary conditions,

$$\frac{\partial T_f}{\partial Z} = \frac{\partial^2 T_f}{\partial Z^2} = 0$$

and thus T_f satisfies the equation

$$K\left[\frac{\partial^2 T_f}{\partial X^2} + \frac{\partial^2 T_f}{\partial Y^2}\right] + \mu\left[\left(\frac{\partial W}{\partial X}\right)^2 + \left(\frac{\partial W}{\partial Y}\right)^2\right] = 0 \tag{6}$$

The boundary conditions for T* and T_f are

$$T^*\,(\Gamma, Z) = 0$$

$$T^*\,(X,Y,0) = T_o^*\,(X,Y)$$

$$\lim_{Z \to \infty} T^* = 0$$

$$T_f\,(\Gamma, Z) = T_w$$

2.2 NONDIMENSIONALIZED EQUATIONS

The mathematical problem is nondimensionalized based on the width of the rectangular duct, L_1, and on the pressure gradient, dP/dZ, with the following variables and parameters:

$$u = \frac{W}{W_{ref}}$$

$$\theta = \frac{T^*}{T_{ref}}$$

$$\theta_f = \frac{T_f}{T_{ref}}$$

$$W_{ref} = -\frac{L_1^2}{\mu}\frac{dP}{dZ}$$

$$T_{ref} = \frac{\mu W_{ref}}{\rho C_p L_1}$$

$$\alpha = X/L_1$$

$$\beta = Y/L_1$$

$$R = L_2/L_1$$

$$\xi = Z/L_1 \, Pe$$

$$Pe = \frac{\rho W_{ref} C_p L_1}{K} \tag{7}$$

The variables α, β, and ξ are defined on the intervals

$$0 \leq \alpha \leq 1$$

$$0 \leq \beta \leq R$$

$$0 \leq \xi \leq \infty \tag{8}$$

Substitution into Eq. (5) for T* gives

$$u \frac{\partial \theta}{\partial \xi} = \frac{\partial^2 \theta}{\partial \alpha^2} + \frac{\partial^2 \theta}{\partial \beta^2} + Pe^{-2} \frac{\partial^2 \theta}{\partial \xi^2} \tag{9}$$

with the boundary conditions

$$\theta(\Gamma, \xi) = 0$$

$$\theta(\alpha, \beta, 0) = \theta_o(\alpha, \beta)$$

$$\lim_{\xi \to \infty} \theta(\alpha,\beta,\xi) = 0 \tag{10}$$

Substitution into Eq. (6) for T_f gives

$$\frac{\partial^2\theta_f}{\partial\alpha^2} + \frac{\partial^2\theta_f}{\partial\beta^2} = -\dot{P}e\left[\left(\frac{\partial u}{\partial\alpha}\right)^2 + \left(\frac{\partial u}{\partial\beta}\right)^2\right] \tag{11}$$

with the boundary conditions

$$\theta_f(\Gamma,\xi) = \theta_w \tag{12}$$

With the substitution

$$\theta_\eta = \theta_f - \theta_w$$

Equation (11) becomes

$$\frac{\partial^2\theta_\eta}{\partial\alpha^2} + \frac{\partial^2\theta_\eta}{\partial\beta^2} = -\dot{P}e\left[\left(\frac{\partial u}{\partial\alpha}\right)^2 + \left(\frac{\partial u}{\partial\beta}\right)^2\right] \tag{13}$$

with the boundary condition

$$\theta_\eta(\Gamma,\xi) = 0$$

SECTION III
APPROXIMATE SOLUTION OF THE ENERGY EQUATION BY THE METHOD OF B. G. GALERKIN

From Kantorovich (Ref. 17) the solution of an equation of the form $L(v) = 0$, where L is some linear differential operator in two variables and whose solution satisfies homogeneous boundary conditions, may be approximated by the method of B. G. Galerkin by constructing an approximate solution of the equation in the form

$$\overline{v}(X,Y) = \sum_{i=1}^{n} a_i \, \phi_i(X,Y)$$

The $\phi_i(X, Y)(i = 1, 2, ---n)$ is a chosen set of linearly independent functions satisfying the boundary conditions and representing the first n functions of some system of functions $\phi_i(X, Y)(i = 1, 2, ---\infty)$ which is complete in the given region. In order that $\overline{v}(X, Y)$ with $n = \infty$ be the exact solution of the given equation, it is necessary that $L(\overline{v})$ be identically equal to zero. This requirement, with $L(\overline{v})$ considered continuous, is equivalent to the requirement of orthogonality of $L(\overline{v})$ to the functions $\phi_i(X, Y)$. With these conditions the system of n linear equations

$$\iint_D L(\overline{v}(X,Y)) \, \phi_i(X,Y)dX \, dY =$$

$$\iint_D L\left(\sum_{j=1}^{n} a_j \, \phi_j(X,Y)\right)\phi_i(X,Y)dX \, dY = 0$$

$$(i = 1,2,---n)$$

may be solved for the n unknown coefficients, C_j, to complete the approximate solution.

The assumed form of the approximate solution to Eq. (9) will be

$$\theta(\alpha,\beta,\xi) = \sum_{n=1}^{K} \sum_{i=1}^{N} a_i^{(n)} \, \phi_i(\alpha,\beta)e^{-B_n\xi} \tag{14}$$

where the functions $\phi_i(\alpha, \beta)$ satisfy the boundary condition

$$\phi_i(\Gamma) = 0$$

and possess continuous first- and second-order derivatives in α and β in the region D and on the closed boundary, Γ, and the $a_i^{(n)}$'s are constants. The constants $a_i^{(n)}$ can be determined by the B. G. Galerkin method.

Substitution of Eq. (14) into Eq. (9) and following the method of B. G. Galerkin, which involves multiplying the equation by each $\phi_k(\alpha, \beta)$

and integrating over the region, D, yields the linear system of equations:

$$\sum_{i=1}^{N} a_i^{(n)} \left[I_1(k,i) + B_n \, I_2(k,i) + (B_n^2/Pe^2) \, I_3(k,i) \right] = 0 \quad (15)$$

$$k = 1,2,\ldots N$$

where the integrals are given by

$$I_1(k,i) = \int_0^1 \int_0^R \nabla^2 \phi_i(\alpha,\beta) \, \phi_k(\alpha,\beta) \, d\alpha \, d\beta \quad (16)$$

$$I_2(k,i) = \int_0^1 \int_0^R u(\alpha,\beta) \, \phi_i(\alpha,\beta) \, \phi_k(\alpha,\beta) \, d\alpha \, d\beta \quad (17)$$

$$I_3(k,i) = \int_0^1 \int_0^R \phi_i(\alpha,\beta) \, \phi_k(\alpha,\beta) \, d\alpha \, d\beta \quad (18)$$

Since $\phi_i(\alpha, \beta)$ is the known selected set, the integrals may be evaluated either analytically or numerically for a given velocity profile, $u(\alpha, \beta)$, to give Eq. (15) as a system of linear homogeneous equations for the undetermined constants, $a_i^{(n)}$.

For a nontrivial solution to exist for the $a_i^{(n)}$'s, the determinant of the coefficients must be identically zero, that is

$$\mathrm{Det} \left[I_1(k,i) + B_n \, I_2(k,i) + (B_n^2/Pe^2) \, I_3(k,i) \right] = 0 \quad (19)$$

Equation (19) represents a polynomial of at most 2N degree in B_n which must be solved for the eigenvalues (B_n's). For each eigenvalue there will be a set of (N-1) linearly independent equations for the $a_i^{(n)}$'s which may be solved in terms of one of the unknown linear combination constants, $a_j^{(n)}$, which will remain undetermined. Selecting the particular undetermined constant as $a_1^{(n)}$, the linear system of equations becomes

$$\sum_{i=2}^{N} C_i^{(n)} \left[I_1(k,i) + B_n\, I_2(k,i) + (B_n^2/Pe^2)\, I_3(k,i) \right] =$$

$$- \left[I_1(k,1) + B_n\, I_2(k,1) + (B_n^2/Pe^2)\, I_3(k,1) \right]$$

$$k = 2,3, \ldots N \qquad (20)$$

where

$$C_i^{(n)} = a_i^{(n)}/a_1^{(n)} \qquad (i \geq 2)$$

With the solution of the $C_i^{(n)}$'s from the system of Eq. (20), the approximate solution of Eq. (9) becomes

$$\theta(\alpha,\beta,\xi) = \sum_{n=1}^{M} a_1^{(n)} \left[\phi_1(\alpha,\beta) + \sum_{i=2}^{N} C_i^{(n)}\, \phi_i(\alpha,\beta) \right] e^{-B_n \xi} \qquad (21)$$

The constants $a_1^{(n)}$ may then be determined from the inlet boundary conditions given for Eq. (10),

$$\theta(\alpha,\beta,0) = \sum_{n=1}^{M} a_1^{(n)} \left[\phi_1(\alpha,\beta) + \sum_{i=2}^{N} C_i^{(n)}\, \phi_i(\alpha,\beta) \right] = \theta_o(\alpha,\beta) \qquad (22)$$

which specifies the function $\theta_o(\alpha, \beta)$ as a linear combination of known functions.

The constants $a_1^{(n)}$ may now be determined by the application of an extension of the Weirstrass approximation theorem (Ref. 20). Hence, multiplication of Eq. (22) by

$$\left[\phi_1(\alpha,\beta) + \sum_{j=2}^{N} C_j^{(n)}\, \phi_j(\alpha,\beta) \right]$$

and integration over the region D gives

$$\sum_{n=1}^{M} a_1^{(n)} \left[I(\ell,n) \right] = J(\ell), \quad \ell = 1,2,\text{---}M \tag{23}$$

where the integrals in Eq. (23) are specified as

$$I(\ell,n) = \int_0^1 \int_0^R \left[\phi_1(\alpha,\beta) + \sum_{i=2}^{N} C_i^{(n)} \phi_i(\alpha,\beta) \right] \left[\phi_1(\alpha,\beta) \right.$$

$$\left. + \sum_{j=2}^{N} C_j^{(n)} \phi_j(\alpha,\beta) \right] d\alpha \, d\beta \tag{24}$$

and

$$J(\ell) = \int_0^1 \int_0^R \theta_0(\alpha,\beta) \left[\phi_1(\alpha,\beta) + \sum_{j=2}^{N} C_j^{(n)} \phi_j(\alpha,\beta) \right] d\alpha \, d\beta \tag{25}$$

Since the $C_i^{(n)}$'s are determined from Eq. (20), Eq. (23) represents a nonhomogeneous system of M linear equations in M unknowns, $a_1^{(n)}$, which has a unique solution for $|\theta_0(\alpha,\beta)| > 0$. For the thermal entrance region problem to exist, the condition is satisfied since $\theta_0(\alpha,\beta)$ represents the difference between the duct inlet temperature and the fully developed temperature profiles. Therefore, the $a_1^{(n)}$'s can be determined to give a complete approximate solution of the thermal entrance region problem.

SECTION IV
NUMERICAL RESULTS

4.1 NUMERICAL PROCEDURE

The general method, developed in the previous section for Eq. (9), was used to approximate the solution of the thermal entrance region heat transfer for rectangular ducts of aspect ratio 1, 2, 5, and 10 and for Peclet numbers, based on the pressure gradient and the velocity nondimensionalization parameter, of 10, 100, 300, and 1000 for each aspect ratio.

The uniform wall temperature boundary condition and the resulting symmetry about the duct centerlines allowed the selection of the functions $\phi_i(\alpha, \beta)$ to be of the form

$$\phi_i(\alpha,\beta) = \sin p\,\pi\alpha \sin q\,\pi\beta/R \tag{26}$$

where p and q are pairs of odd integers corresponding to i.

For the special case of the square duct, the extra symmetry of the equal wall lengths allowed the functions, $\phi_i(\alpha, \beta)$, to be written in the double form

$$\phi_i(\alpha,\beta) = \sin p\,\pi\alpha \sin q\,\pi\beta + \sin q\,\pi\alpha \sin p\,\pi\beta \tag{27}$$

where p and q again are odd integers corresponding to i. This, in turn, allowed a smaller number of functions to be used for the square duct solution with a consequent decrease in computation time required. It was found, by solving specific cases with different numbers of functions, that ten functions for the square cases and 15 functions for the nonsquare cases gave satisfactory convergence.

The equation of motion (Eq. (1)) was solved in terms of a double Fourier series for the velocity profile. The solution is given as Eq. (I-14) in Appendix I. The integrals, $I_1(k, i)$, $I_2(k, i)$, and $I_3(k, i)$, were integrated analytically. The integration is given in Appendix III. This allowed a general expression for the elements of the determinant (Eq. (19)) to be determined in terms of the indices k and i.

The determinant was found to be diagonally dominant, and the roots of the determinant, B_n's, were evaluated by use of the maximum diagonal pivot Gaussian reduction technique with the final iteration by the method of reguli-falsi. The B_n's were sufficiently iterated to guarantee an accuracy of eight significant digits.

Table I gives the integer pairs (p, q) for the functions $\phi_i(\alpha, \beta)$. It should be noted that this is a triangular truncation of the approximating series. It was found that this gave quicker convergence than a square truncation for this type of solution where the coefficients are determined from a set of linear nonhomogeneous equations. However, this is not necessarily true for a series such as that found for the velocity profile solution, Eq. (I-14), in Appendix I.

TABLE I
INTEGER PAIRS (p, q) FOR THE FUNCTIONS $\phi_i(\alpha, \beta)$

i	R ≠ 1 Pe = 10, 100, 300, 1000	R = 1 Pe = 100, 300, 1000	R = 1 Pe = 1, 10
1	1,1	1,1	1,1
2	1,3	1,3	1,3
3	1,5	1,5	1,5
4	1,7	1,7	1,7
5	1,9	1,9	1,9
6	3,1	1,11	1,11
7	3,3	3,3	3,3
8	3,5	3,5	3,5
9	3,7	3,7	3,7
10	5,1	5,5	3,9
11	5,3		
12	5,5		
13	7,1		
14	7,3		
15	9,1		

Tables II through VII give the eigenvalues for the cases of aspect ratios 1, 2, 5, and 10 and Peclet numbers 10, 100, 300, and 1000. Also included are the cases of aspect ratio 1.5 and Peclet number 1000 and aspect ratio 1 and Peclet number 1.

TABLE II
EIGENVALUES AND INTEGER PAIRS FOR ASPECT RATIO OF 1 AND Pe OF 1 AND 10

p	q	B_n	
		Pe = 1	Pe = 10
1	1	4.41418×10^0	4.16395×10^1
1	3	9.91052×10^0	9.69653×10^1
3	3	1.33097×10^1	1.31405×10^2
1	5	1.59962×10^1	1.57925×10^2
3	5	1.82987×10^1	1.81228×10^2
1	7	2.21918×10^1	2.19889×10^2
3	7	2.39072×10^1	2.37424×10^2
1	9	2.84257×10^1	2.82231×10^2
3	9	2.97854×10^1	2.96222×10^2
1	11	3.46775×10^1	3.44763×10^2

TABLE III
EIGENVALUES AND INTEGER PAIRS FOR ASPECT RATIO OF 1 AND Pe OF 100, 300, AND 1000

p	q	B_n		
		Pe = 100	Pe = 300	Pe = 1000
1	1	2.40433×10^2	3.19609×10^2	3.36964×10^2
1	3	7.80820×10^2	1.51035×10^3	1.94644×10^3
3	3	1.15619×10^3	2.62382×10^3	4.09313×10^3
1	5	1.38884×10^3	3.15950×10^3	4.99693×10^3
3	5	1.64573×10^3	4.01269×10^3	7.34183×10^3
1	7	2.00434×10^3	4.91449×10^3	9.07017×10^3
5	5	2.05702×10^3	5.40188×10^3	1.21179×10^4
3	7	2.22453×10^3	5.84892×10^3	1.32607×10^4
1	9	2.62970×10^3	6.76701×10^3	1.40835×10^4
1	11	3.25869×10^3	8.72051×10^3	2.10690×10^4

TABLE IV
EIGENVALUES AND INTEGER PAIRS FOR ASPECT RATIO OF 1.5 AND Pe OF 1000

p	q	B_n Pe = 1000
1	1	1.76879×10^2
1	3	7.25395×10^2
3	1	1.38991×10^3
1	5	1.79159×10^3
3	3	2.27422×10^3
3	5	3.76529×10^3
1	7	3.82280×10^3
5	1	4.20636×10^3
5	3	5.80292×10^3
7	1	8.21247×10^3

TABLE V
EIGENVALUES AND INTEGER PAIRS FOR ASPECT RATIO OF 2 AND Pe OF 10, 100, 300, AND 1000

p	q	B_n			
		Pe = 10	Pe = 100	Pe = 300	Pe = 1000
1	1	3.08423×10^1	1.18467×10^2	1.31451×10^2	1.33316×10^2
1	3	5.28847×10^1	2.96333×10^2	3.85064×10^2	4.03407×10^2
1	5	8.09030×10^1	5.47520×10^2	8.35539×10^2	9.24158×10^2
3	1	9.19682×10^1	6.54024×10^2	1.05383×10^3	1.19538×10^3
3	3	1.02318×10^2	7.84682×10^2	1.38913×10^3	1.66049×10^3
1	7	1.10675×10^2	8.27962×10^2	1.43639×10^3	1.70509×10^3
3	5	1.19715×10^2	9.64982×10^2	1.87316×10^3	2.42366×10^3
1	9	1.41147×10^2	1.14109×10^3	2.31374×10^3	3.21220×10^3
3	7	1.41921×10^2	1.20110×10^3	2.48366×10^3	3.27438×10^3
5	1	1.54327×10^2	1.25707×10^3	2.63597×10^3	4.02052×10^3
5	3	1.60993×10^2	1.36135×10^3	2.87559×10^3	4.14845×10^3
5	5	1.72741×10^2	1.51392×10^3	3.53555×10^3	6.01251×10^3
7	1	2.16938×10^2	1.87356×10^3	4.14751×10^3	6.36712×10^3
7	3	2.21957×10^2	1.98928×10^3	4.80427×10^3	8.88796×10^3
9	1	2.79692×10^2	2.51698×10^3	6.16419×10^3	1.20650×10^3

TABLE VI
EIGENVALUES AND INTEGER PAIRS FOR ASPECT RATIO OF 5 AND Pe OF 10, 100, 300, AND 1000

p	q	B_n			
		Pe = 10	Pe = 100	Pe = 300	Pe = 1000
1	1	2.71323×10^1	8.82645×10^1	9.45499×10^1	9.53839×10^1
1	3	3.19397×10^1	1.18039×10^2	1.29705×10^2	1.31344×10^2
1	5	3.98096×10^1	1.70549×10^2	1.95330×10^2	1.99152×10^2
1	7	4.94473×10^1	2.41554×10^2	2.91486×10^2	3.00176×10^2
1	9	6.01129×10^1	3.35243×10^2	4.41311×10^2	4.64455×10^2
3	1	9.03114×10^1	6.10023×10^2	9.30171×10^2	1.02871×10^3
3	3	9.21879×10^1	6.34871×10^2	9.84766×10^2	1.09692×10^3
3	5	9.55470×10^1	6.75797×10^2	1.08452×10^3	1.22914×10^3
3	7	1.00300×10^2	7.47551×10^2	1.31498×10^3	1.57715×10^3
5	1	1.53107×10^2	1.21204×10^3	2.29434×10^3	2.90775×10^3
5	3	1.54326×10^2	1.23440×10^3	2.38673×10^3	3.09339×10^3
5	5	1.56509×10^2	1.29307×10^3	2.67877×10^3	3.77781×10^3
7	1	2.15909×10^2	1.82828×10^3	3.90311×10^3	5.66938×10^3
7	3	2.16910×10^2	1.87311×10^3	4.24036×10^3	6.97333×10^3
9	1	2.78759×10^2	2.46842×10^3	5.89038×10^3	1.09241×10^4

TABLE VII
EIGENVALUES AND INTEGER PAIRS FOR ASPECT RATIO OF 10 AND Pe OF 10, 100, 300, AND 1000

p	q	B_n			
		Pe = 10	Pe = 100	Pe = 300	Pe = 1000
1	1	2.66084×10^1	8.49501×10^1	9.07163×10^1	9.14761×10^1
1	3	2.78789×10^1	9.20748×10^1	9.88936×10^1	9.98038×10^1
1	5	3.02449×10^1	1.05706×10^2	1.14767×10^2	1.16006×10^2
1	7	3.34911×10^1	1.25430×10^2	1.38313×10^2	1.40138×10^2
1	9	3.74307×10^1	1.52746×10^2	1.72788×10^2	1.75807×10^2
3	1	9.01022×10^1	6.06540×10^2	9.22175×10^2	1.01866×10^3
3	3	9.05687×10^1	6.11968×10^2	9.33374×10^2	1.03235×10^3
3	5	9.14621×10^1	6.23653×10^2	9.62644×10^2	1.07083×10^3
3	7	9.27769×10^1	6.51246×10^2	1.04882×10^3	1.19224×10^3
5	1	1.52964×10^2	1.20916×10^3	2.28500×10^3	2.89177×10^3
5	3	1.53264×10^2	1.21497×10^3	2.31009×10^3	2.94085×10^3
5	5	1.53871×10^2	1.23987×10^3	2.46839×10^3	3.33297×10^3
7	1	2.15795×10^2	1.82484×10^3	3.88155×10^3	5.60739×10^3
7	3	2.16058×10^2	1.84707×10^3	4.11711×10^3	6.59961×10^3
9	1	2.78647×10^2	2.46319×10^3	5.86196×10^3	1.08121×10^4

For the lower Peclet numbers the integrals $I_1(k, i)$ and $I_3(k, i)$, as specified by Eqs. (16) and (18), respectively, dominate the determinant to the extent that two integer pairs with the same value of the sum $(p^2 + q^2/R^2)$ produced nearly equal eigenvalues. For the cases of aspect ratio 1 and Peclet numbers 1 and 10, the eigenvalues produced by the pairs (1, 7) and (5, 5) differed only in the fifth significant digit. This was close enough to produce difficulty in evaluating the nonhomogeneous system, Eq. (23), and as shown in Table I for these cases, the integer pair (3, 9) was substituted for the pair (5, 5).

It was found that each function $\phi_i(\alpha, \beta)$ produced an eigenvalue, and for the lower Peclet numbers it was possible to identify which function generated which eigenvalue on the basis of the value of $p^2 + q^2/R^2$. In Tables II through VII the eigenvalues are listed in ascending order. Listed with the eigenvalues are the integer pairs which generated them.

The inlet temperature boundary condition $\theta_o(\alpha, \beta)$, which was defined as the difference between the temperature at the duct inlet and the final developed viscous temperature profile, was written as

$$\theta_o(\alpha,\beta) = \Delta\theta_T - \theta_\eta(\alpha,\beta) \tag{28}$$

where $\Delta\theta_T$ is the temperature difference between the constant temperature inlet condition and the uniform wall temperature, and $\theta_\eta(\alpha, \beta)$ is the solution of the transformed viscous dissipation equation given as Eq. (II-10) in Appendix II.

The constants $a_1^{(n)}$ from Eq. (23) were solved for by the maximum diagonal pivot Gaussian reduction technique. With separation of the inlet boundary conditions into nonviscous and viscous dissipation components, two sets of constants, $a_{1\lambda}^{(n)}$ and $a_{1\eta}^{(n)}$, were solved for during the same matrix reduction. This was accomplished by writing Eq. (25) for $J(\ell)$ as

$$J(\ell) = J_\lambda(\ell) - J_\eta(\ell) \tag{29}$$

where

$$J_\lambda(\ell) = \int_0^1 \int_0^R \left[\phi_I(\alpha,\beta) + \sum_{J=2}^{N} C_J^{(n)} \phi_J(\alpha,\beta) \right] d\alpha \, d\beta \tag{30}$$

and

$$J_\eta(\ell) = \int_0^1 \int_0^R \theta_\eta(\alpha,\beta) \left[\phi_1(\alpha,\beta) + \sum_{j=2}^{N} C_j^{(n)} \phi_j(\alpha,\beta) \right] d\alpha \, d\beta$$

$$= \int_0^1 \int_0^R \sum_{s=1,3,5..}^{\infty} \sum_{t=1,3,5..}^{\infty} \mathcal{F}_s(s,t) \sin s\pi\alpha \sin t\pi\beta/R$$

$$\left[\phi_1(\alpha,\beta) + \sum_{j=2}^{N} C_j^{(n)} \phi_j(\alpha,\beta) \right] d\alpha \, d\beta \qquad (31)$$

and by specifying $J_\lambda(\ell)$ and $J_\eta(\ell)$ as separate column vectors in the matrix solution for $a_{1\lambda}^{(n)}$ and $a_{1\eta}^{(n)}$. Given in Appendix IV as Eqs. (IV-1) through (IV-9) are the integrated solution forms for Eq. (24) for $I(\ell, n)$ and Eqs. (30) and (31) for $J_\lambda(\ell)$ and $J_\eta(\ell)$.

The temperature solutions now become, for the nonviscous component,

$$\theta_\lambda(\alpha,\beta,\xi) = \Delta\theta_T \sum_{n=1}^{M} a_{1\lambda}^{(n)} \left[\phi_1(\alpha,\beta) \right.$$

$$\left. + \sum_{l=2}^{N} C_l^{(n)} \phi_l(\alpha,\beta) \right] e^{-B_n \xi} \qquad (32)$$

and for the viscous component,

$$\theta_\eta(\alpha,\beta,\xi) = Pe \sum_{n=1}^{M} a_{1\eta}^{(n)} \left[\phi_1(\alpha,\beta) + \right.$$

$$\sum_{i=2}^{N} C_i^{(n)} \quad \phi_i(\alpha,\beta) \Bigg] e^{-B_n \xi} \tag{33}$$

4.2 RESULTS

Of primary interest in this study are the bulk mean temperature, θ_m, as specified by Eqs. (IV-10) through (IV-14) in Appendix IV, the Nusselt numbers, Nu, as specified by Eqs. (IV-20) and (IV-21) in Appendix IV, and the ratio of the long-to-short wall heat-transfer rates, Q_β/Q_α, as may be determined from Eqs. (IV-15) through (IV-19) in Appendix IV.

Figure 2 gives the nondimensionalized mean velocity, u_m, as evaluated from Eq. (I-17), as a function of aspect ratio. It is this mean velocity which must be multiplied by the Peclet number used in this study to obtain the more standard Peclet number.

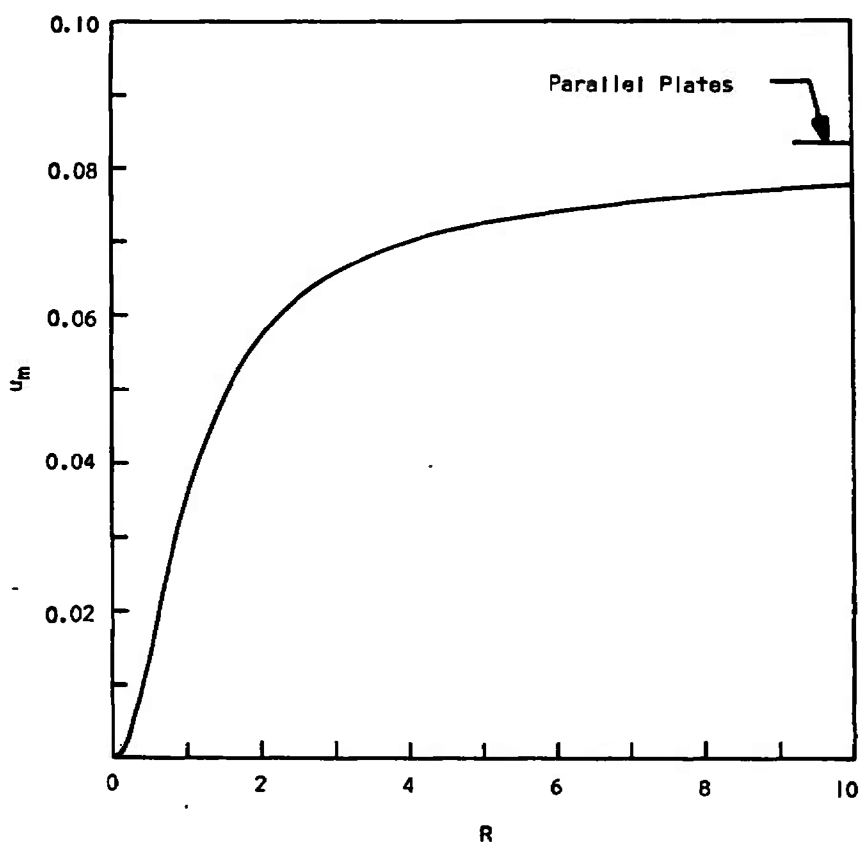

Fig. 2 Mean Velocity versus Aspect Ratio

Figure 3 gives a comparison of the nonviscous component of the bulk mean temperature for aspect ratio of 10 and Peclet numbers of 1000 and 100 to that given by Ref. 2 for the parallel plate solution with no axial condition. The parameter, Pe', is the Peclet number based on the mean velocity for parallel plate flow as used by Ref. 2. Figure 4 gives a comparison of the nonviscous component of the bulk mean temperature for aspect ratio of 1 and Peclet numbers of 1000 and 100 to that given by Ref. 15 for no axial conduction. For aspect ratio of 1, the Peclet number of 1000 as used in this study corresponds to a Peclet number, based on the mean velocity, of 35:. The data from Ref. 15 should be applicable for Peclet numbers, based on the mean velocity, of 100 and above. It may be seen from this figure that ignoring axial conduction for low Peclet numbers can give considerable error.

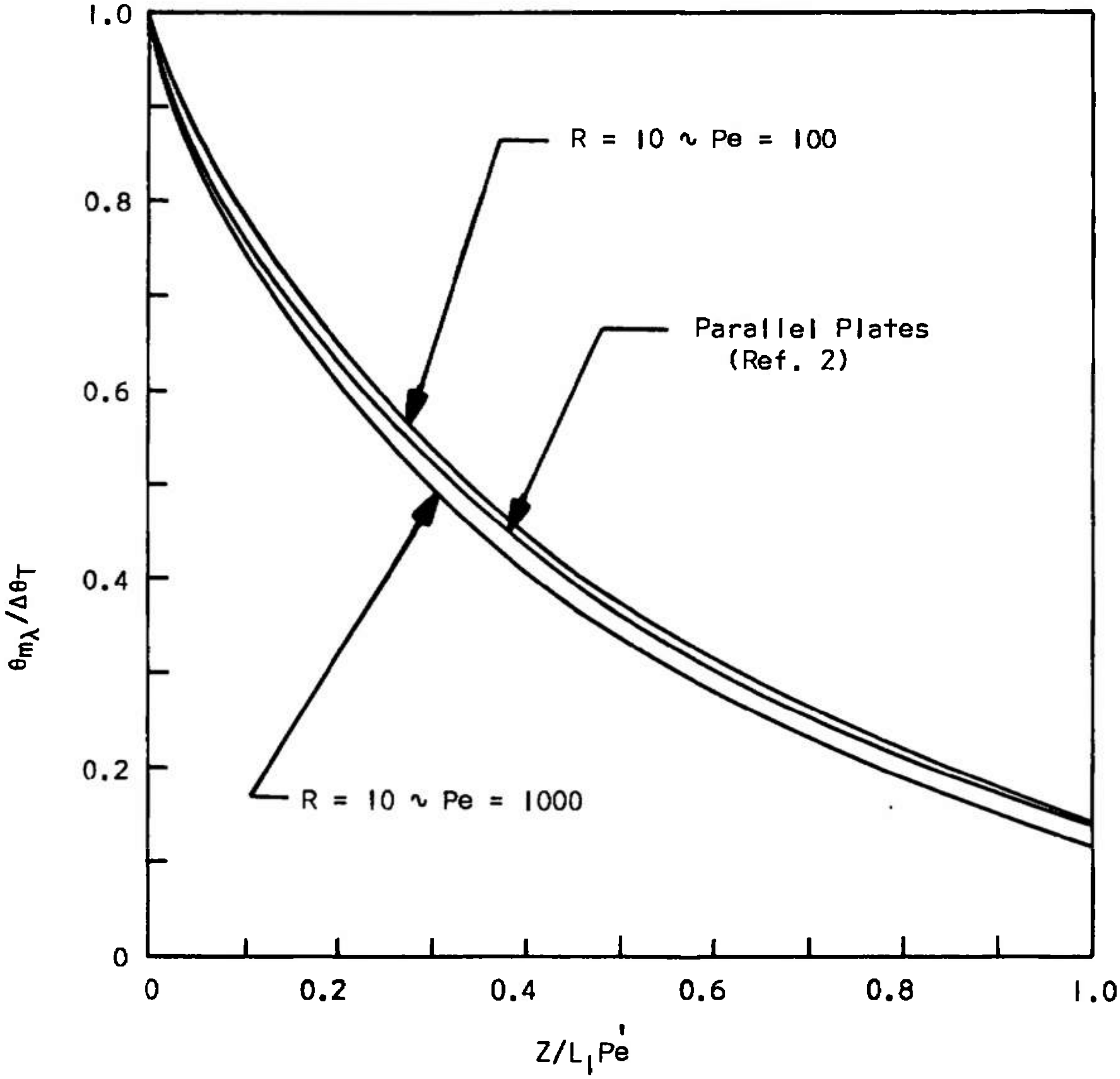

Fig. 3 Bulk Mean Temperature for Aspect Ratio of 10 and Parallel Plates

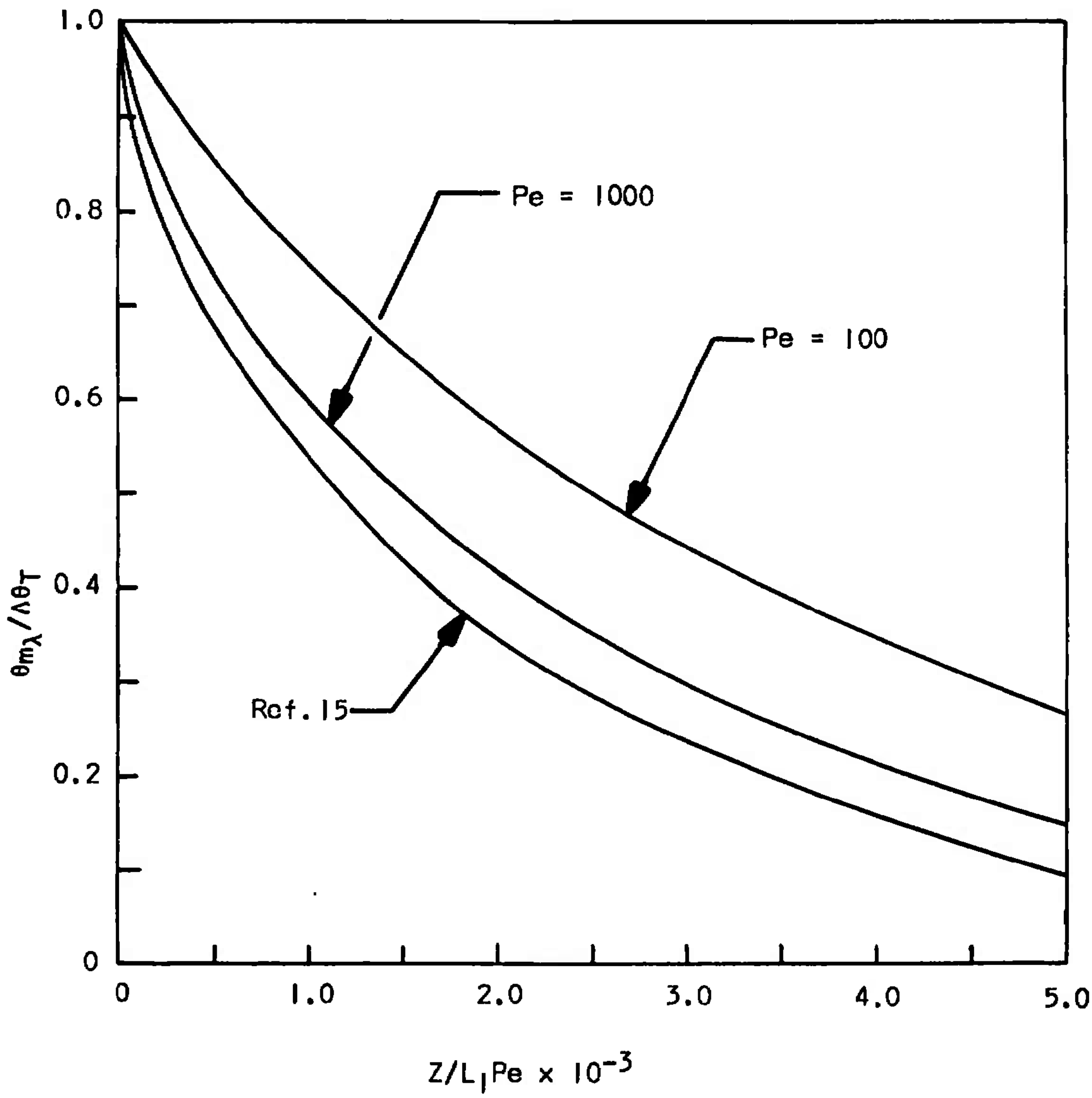

Fig. 4 Bulk Mean Temperature for Aspect Ratio of 1

Figures 5 through 8 give the nonviscous component of the bulk mean temperature for aspect ratios of 1, 2, 5, and 10, respectively, with the Peclet number as a parameter. Figures 9 through 12 give the same temperature distribution for Peclet numbers 10, 100, 300, and 1000, respectively, with the aspect ratio as a parameter.

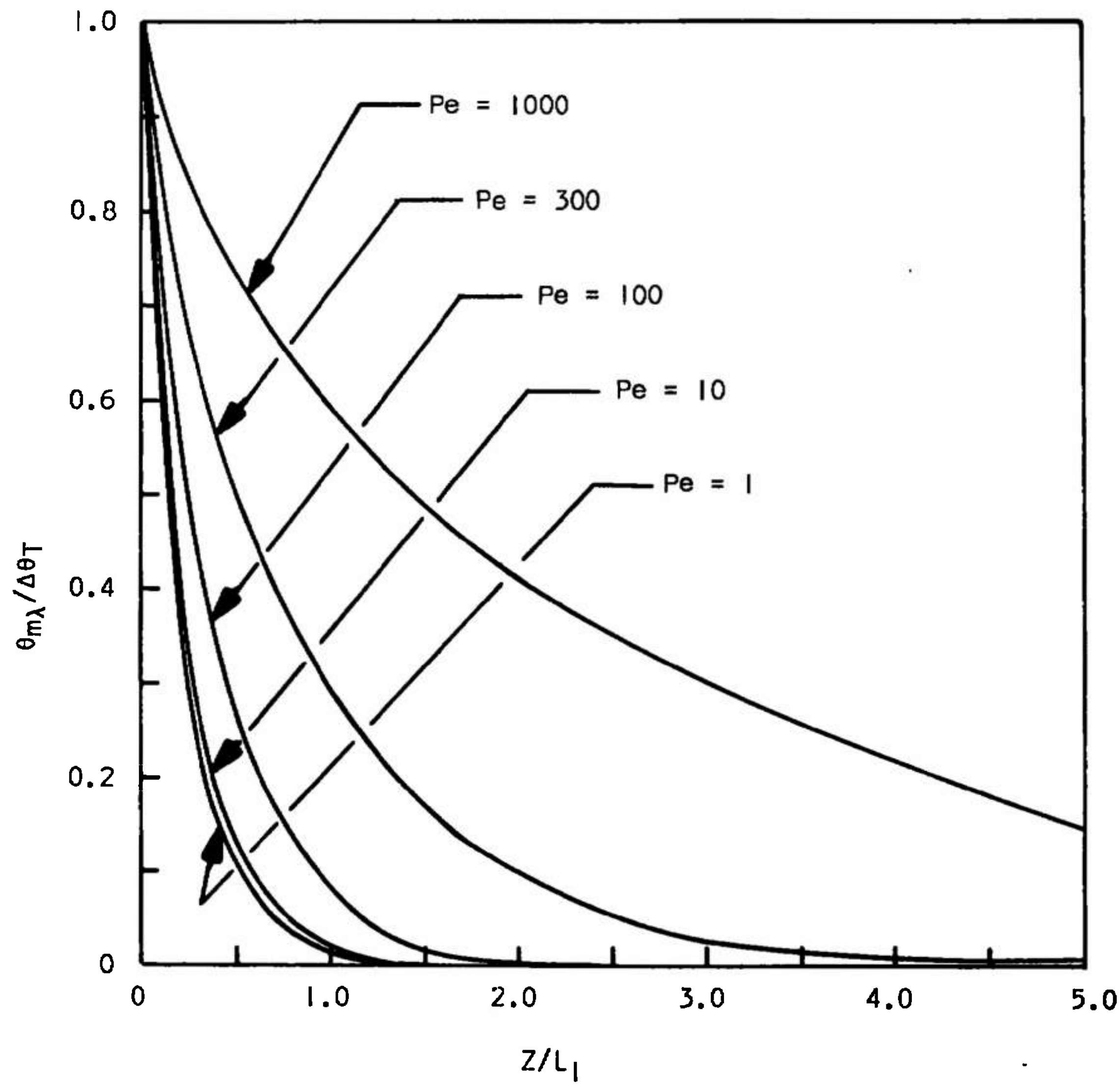

Fig. 5 Bulk Mean Temperature for Aspect Ratio of 1 and Various Peclet Numbers

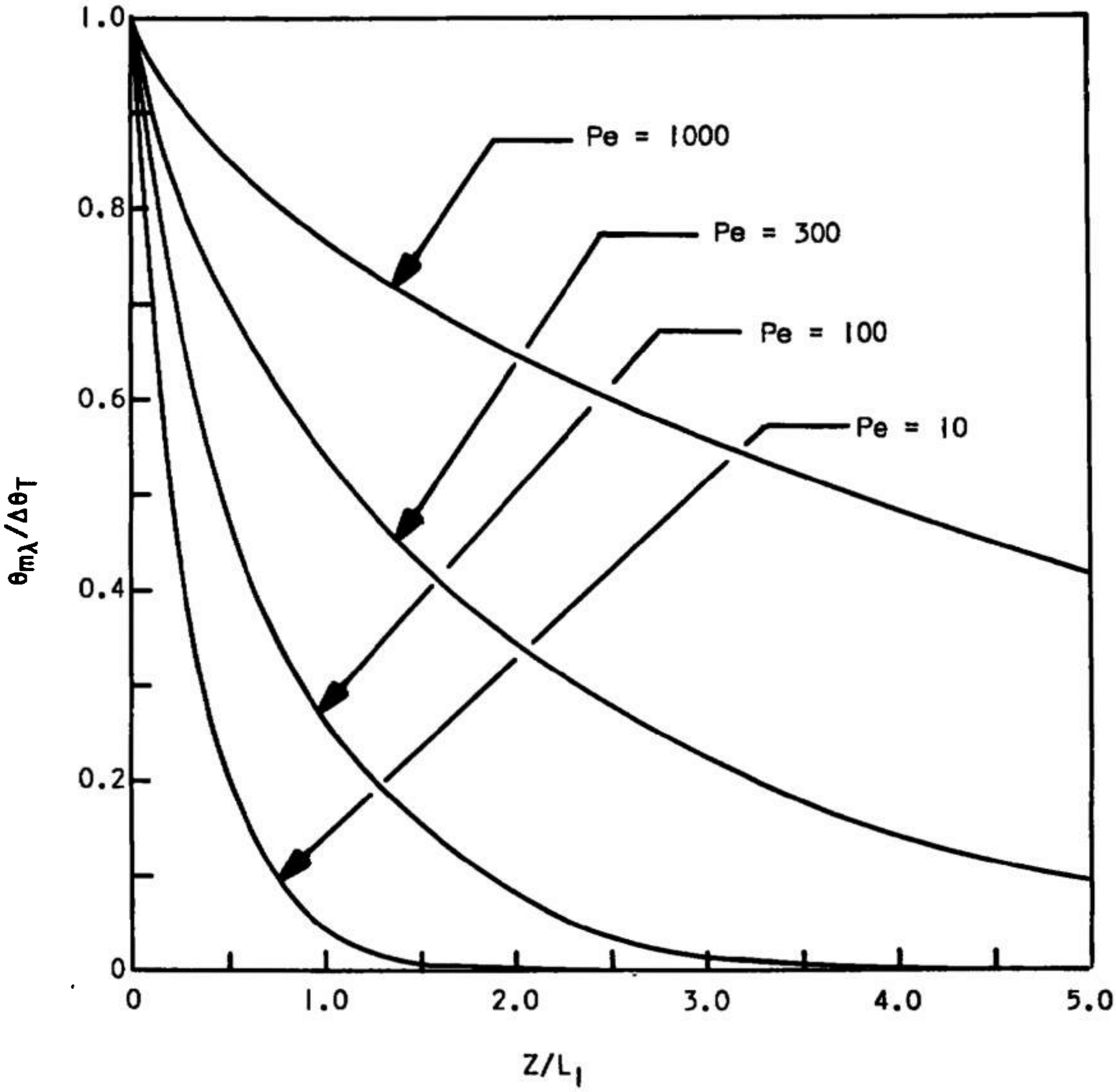

Fig. 6 Bulk Mean Temperature for Aspect Ratio of 2 and Various Peclet Numbers

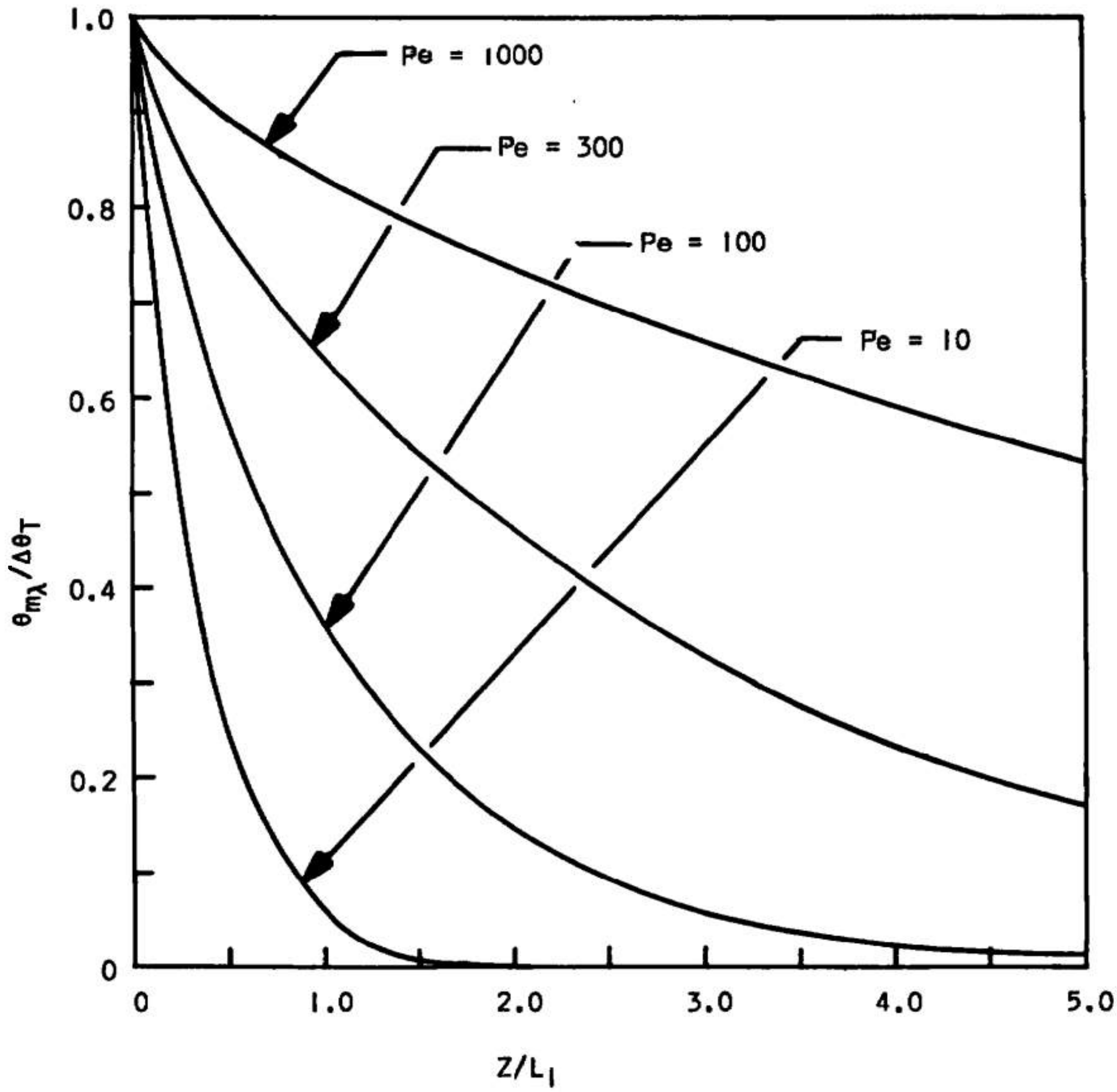

Fig. 7 Bulk Mean Temperature for Aspect Ratio of 5 and Various Peclet Numbers

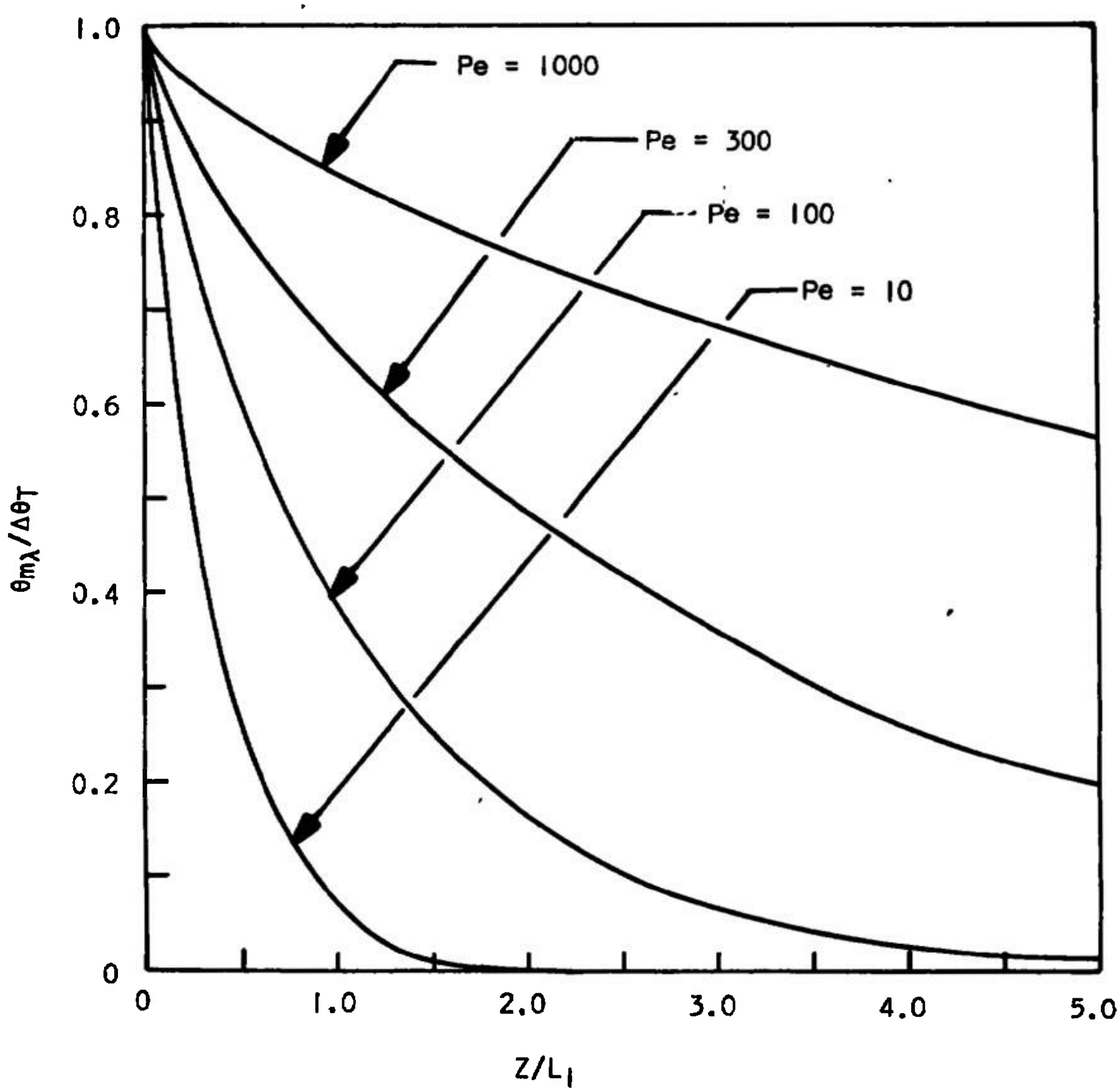

Fig. 8 Bulk Mean Temperature for Aspect Ratio of 10 and Various Peclet Numbers

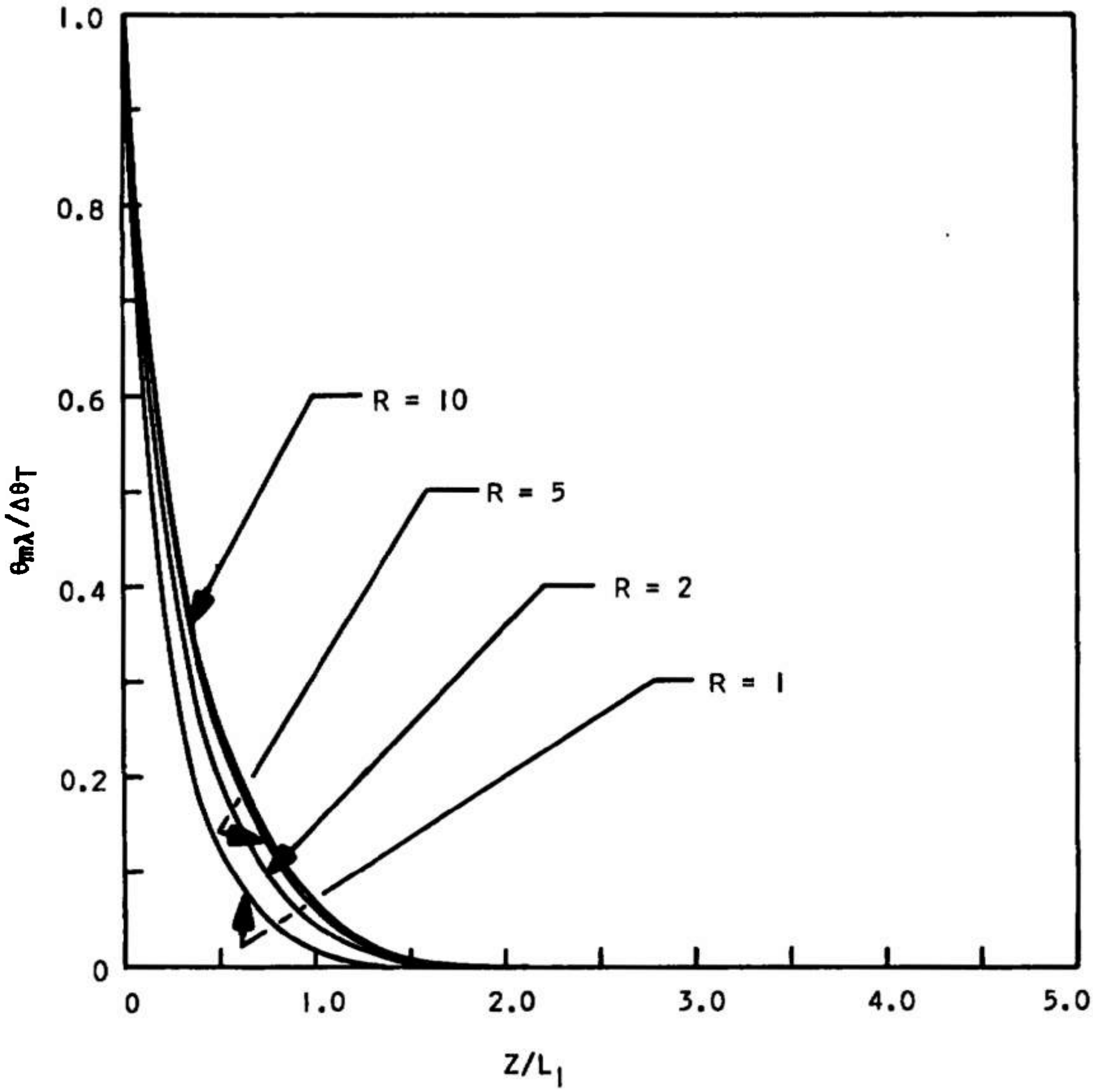

Fig. 9 Bulk Mean Temperature for Pe of 10 and Various Aspect Ratios

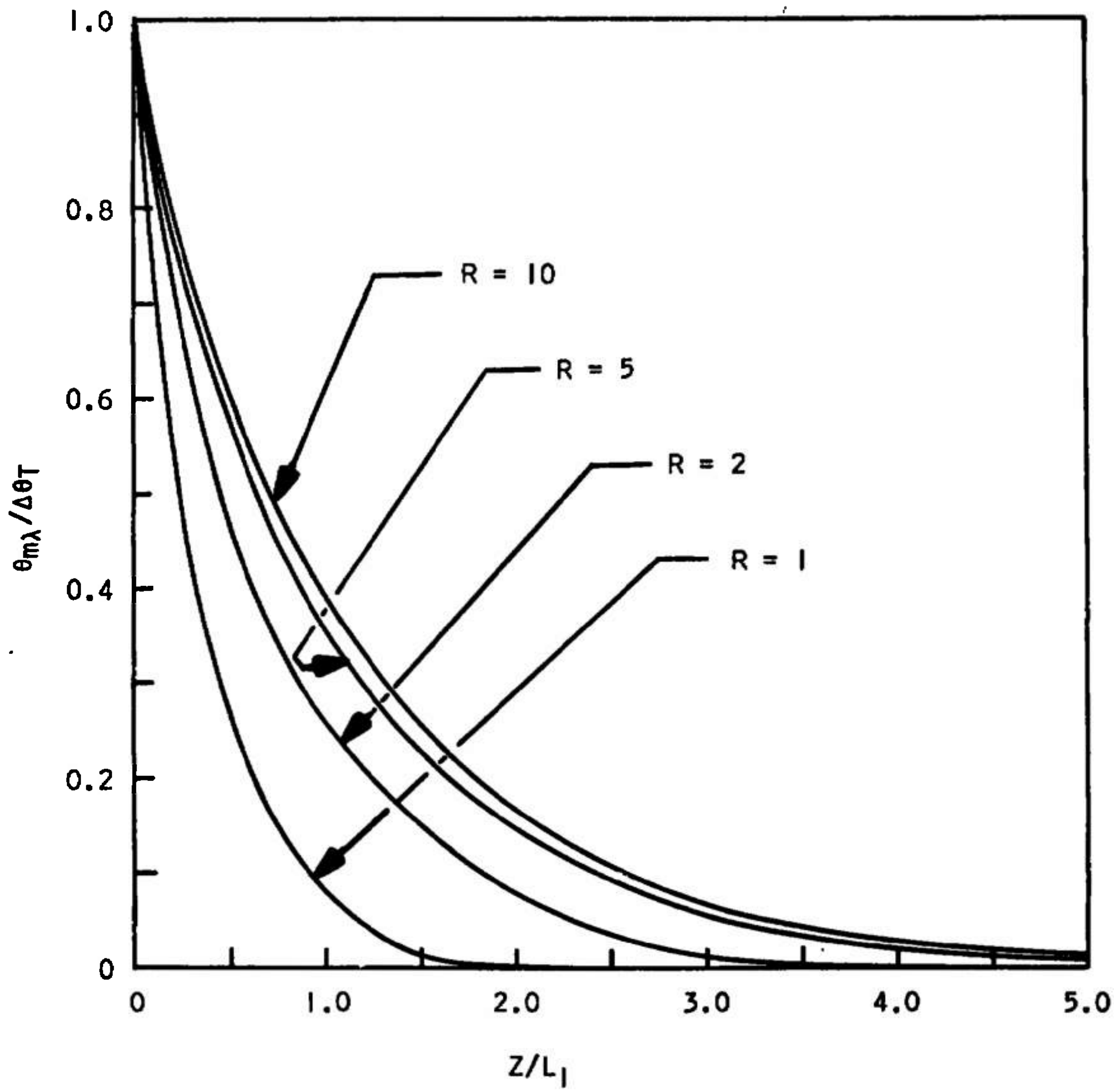

Fig. 10 Bulk Mean Temperature for Pe of 100 and Various Aspect Ratios

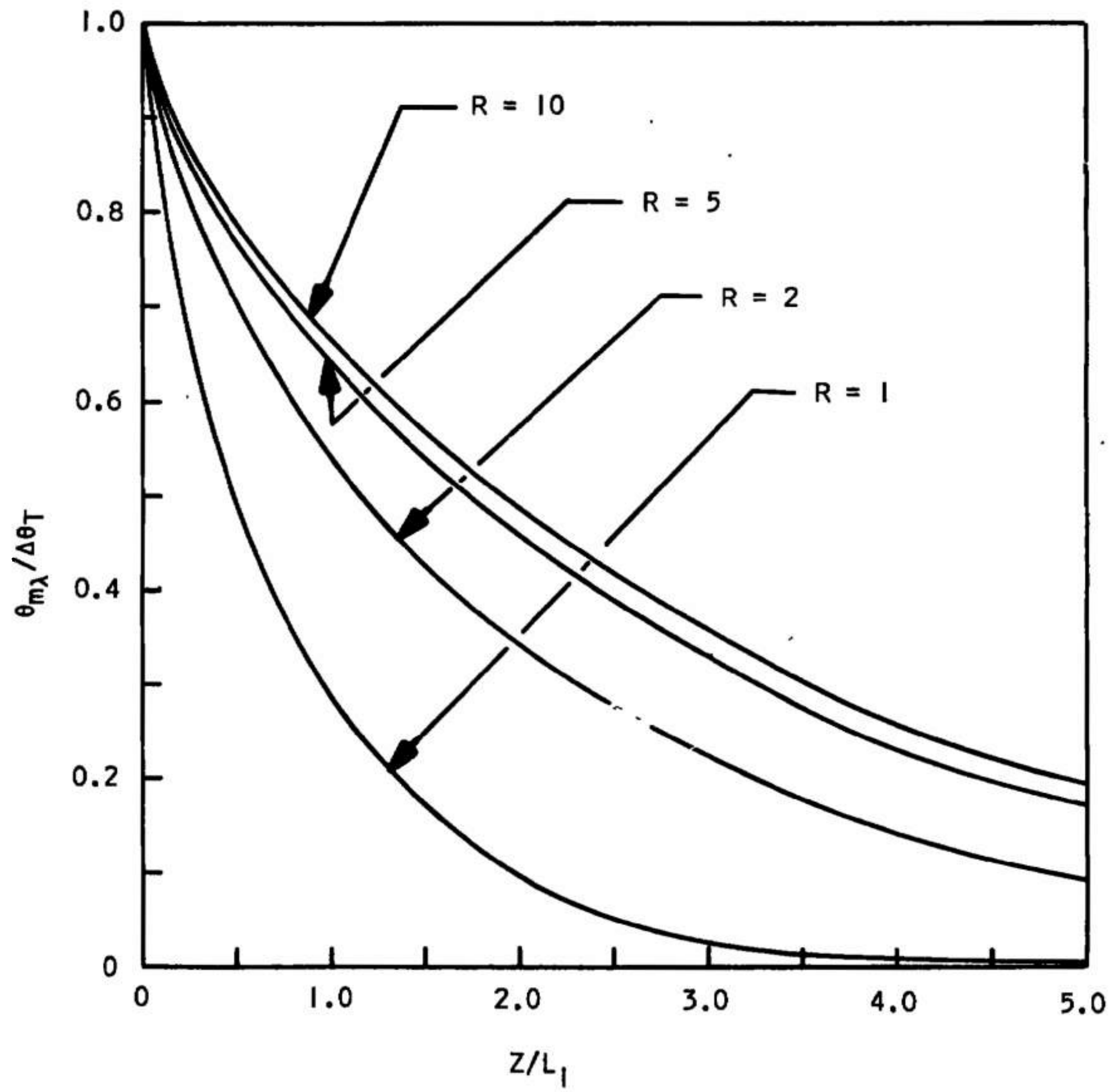

Fig. 11 Bulk Mean Temperature for Pe of 300 and Various Aspect Ratios

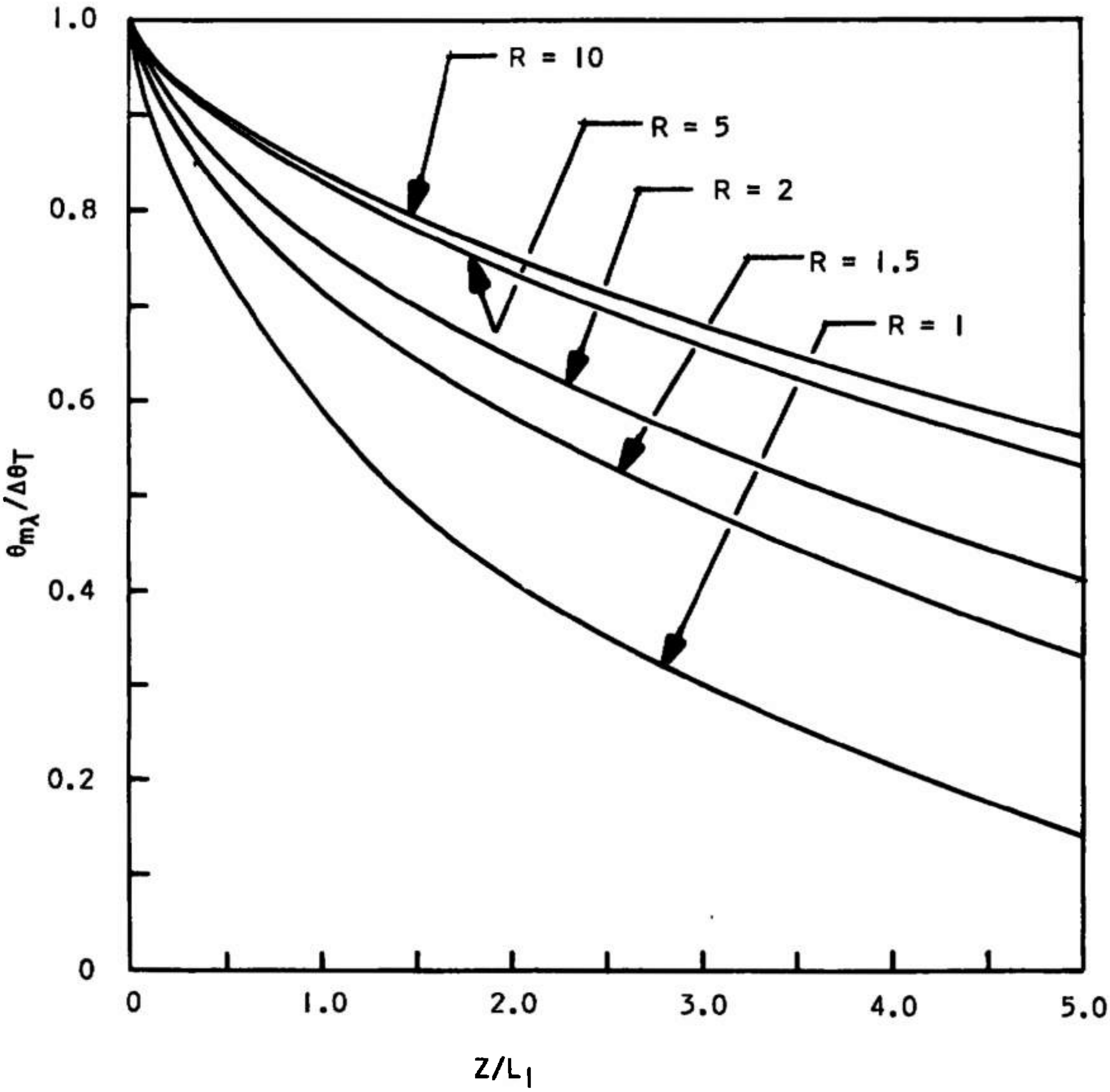

Fig. 12 Bulk Mean Temperature for Pe of 1000 and Various Aspect Ratios

Figure 13 gives the bulk viscous dissipation mean temperature for fully developed flow as a function of aspect ratio. Included is the value for the parallel plate solution.

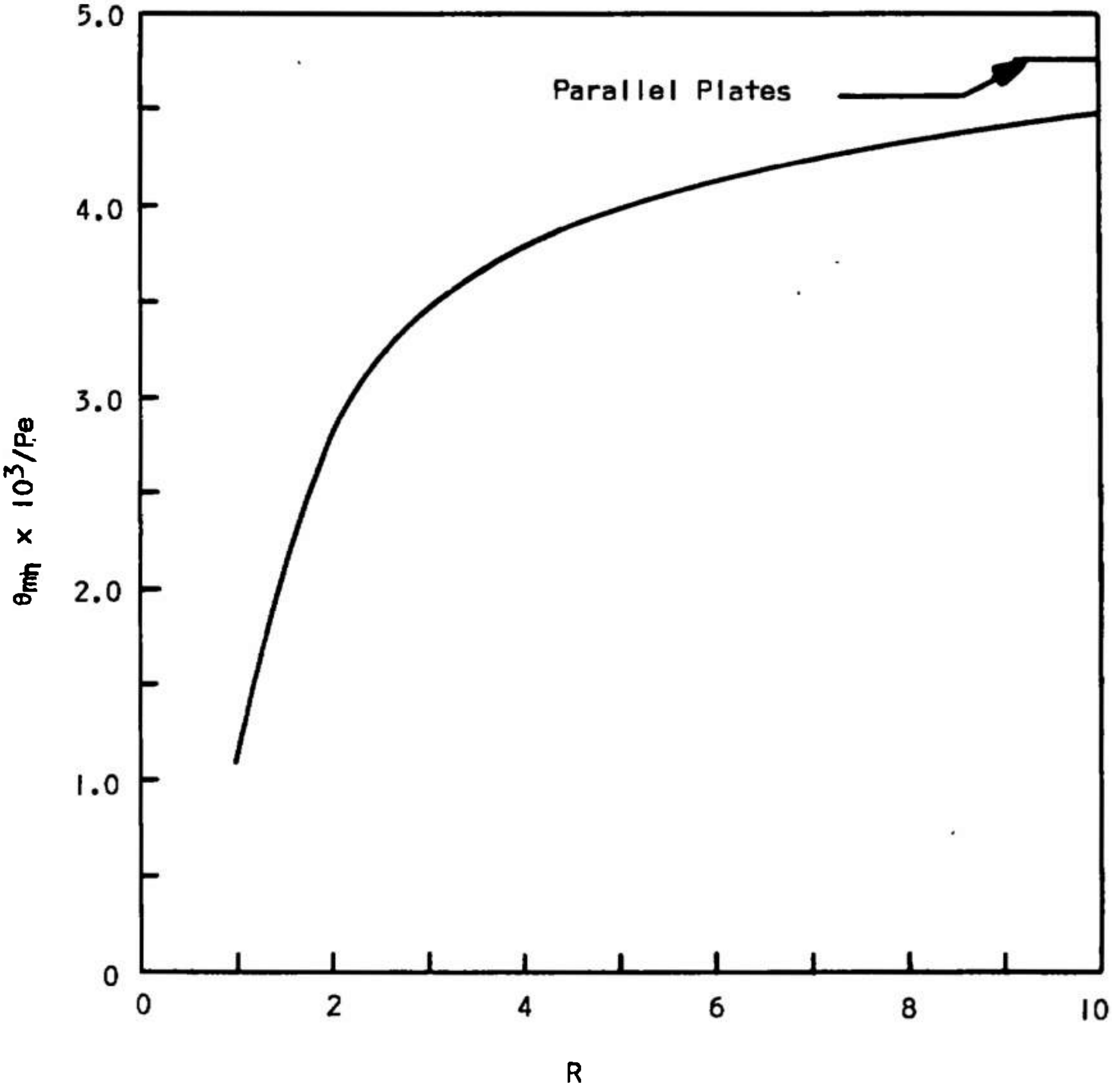

Fig. 13 Bulk Mean Viscous Temperature for Fully Developed Flow versus Aspect Ratio

Figures 14 through 17 give the viscous component of the bulk mean temperature for aspect ratios of 1, 2, 5, and 10, respectively, with the Peclet number as a parameter. Figures 18 through 21 give the same temperature distribution for Peclet numbers of 10, 100, 300, and 1000, respectively, with the aspect ratio as a parameter.

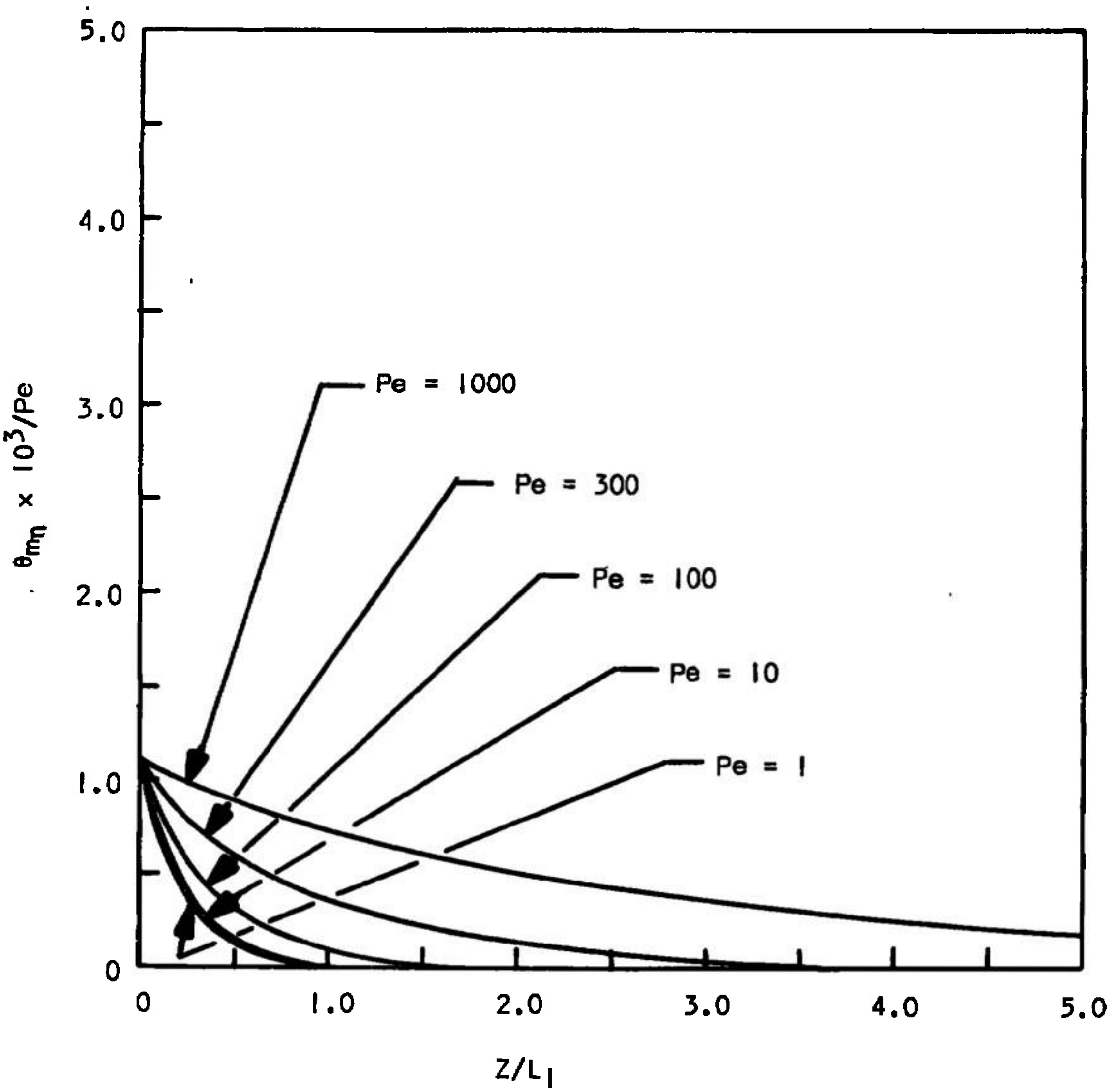

Fig. 14 Bulk Mean Viscous Temperature for Aspect Ratio of 1 and Various Peclet Numbers

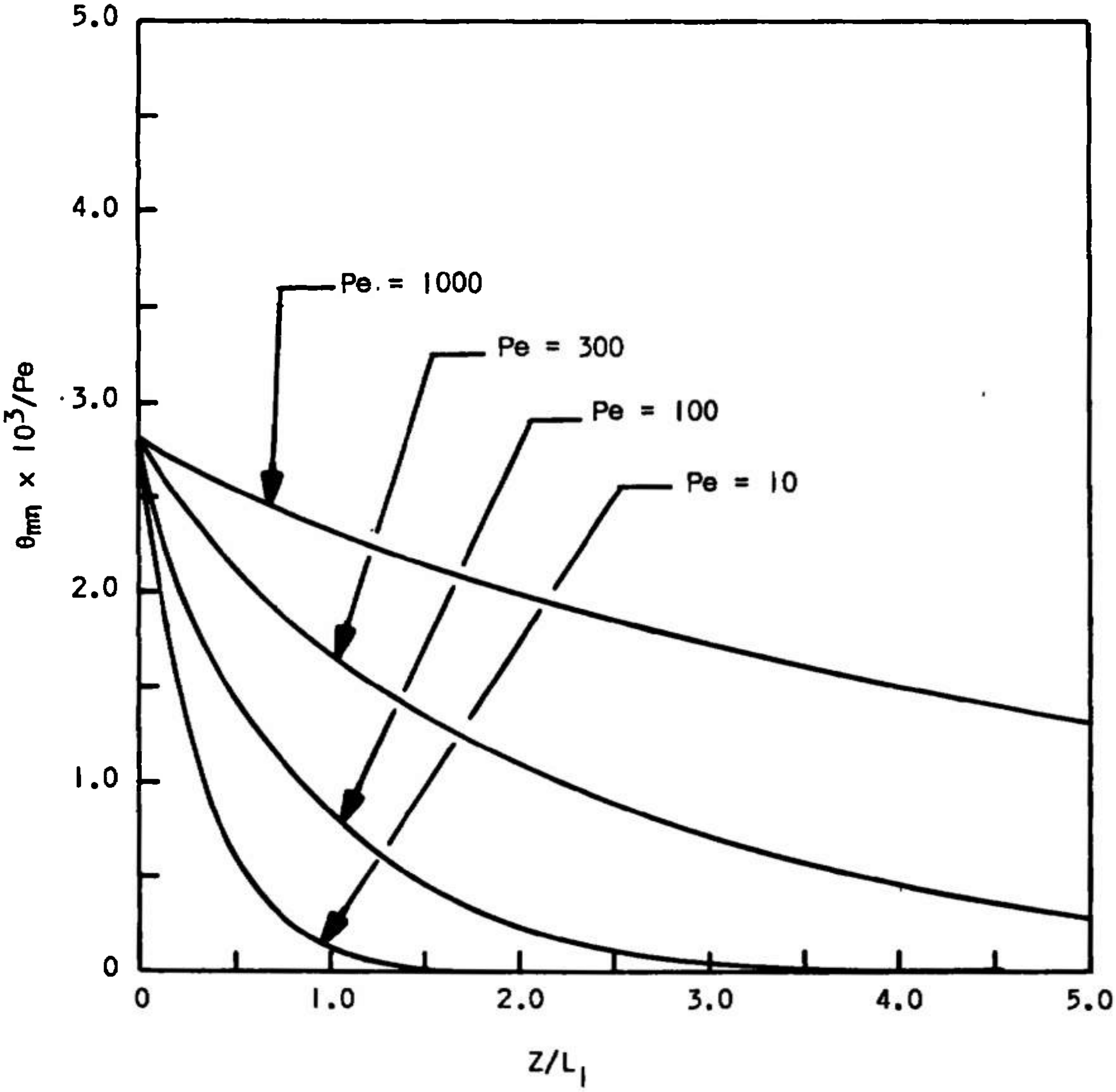

Fig. 15 Bulk Mean Viscous Temperature for Aspect Ratio of 2 and Various Peclet Numbers

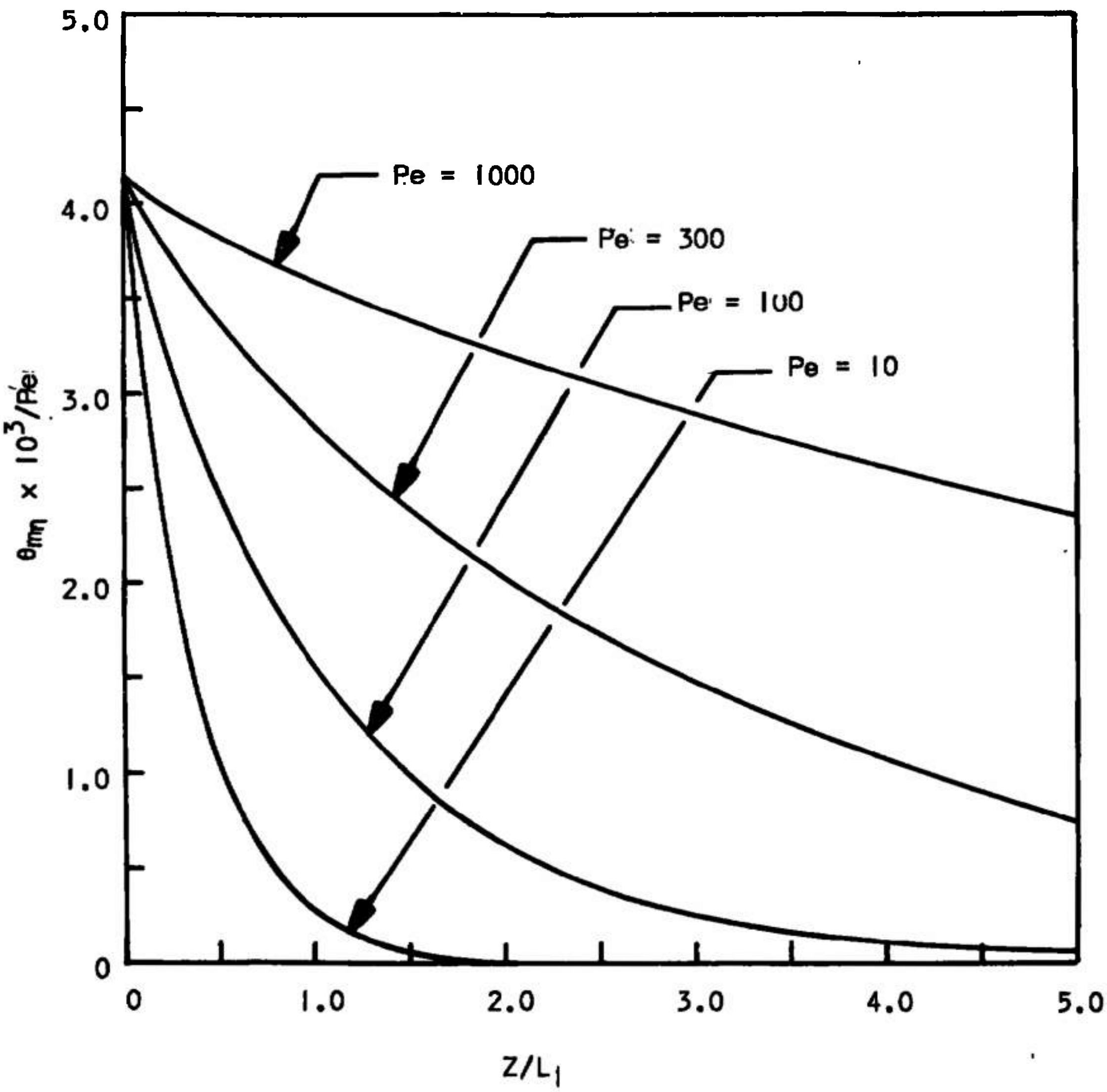

Fig. 16 Bulk Mean Viscous Temperature for Aspect Ratio of 5 and Various Peclet Numbers

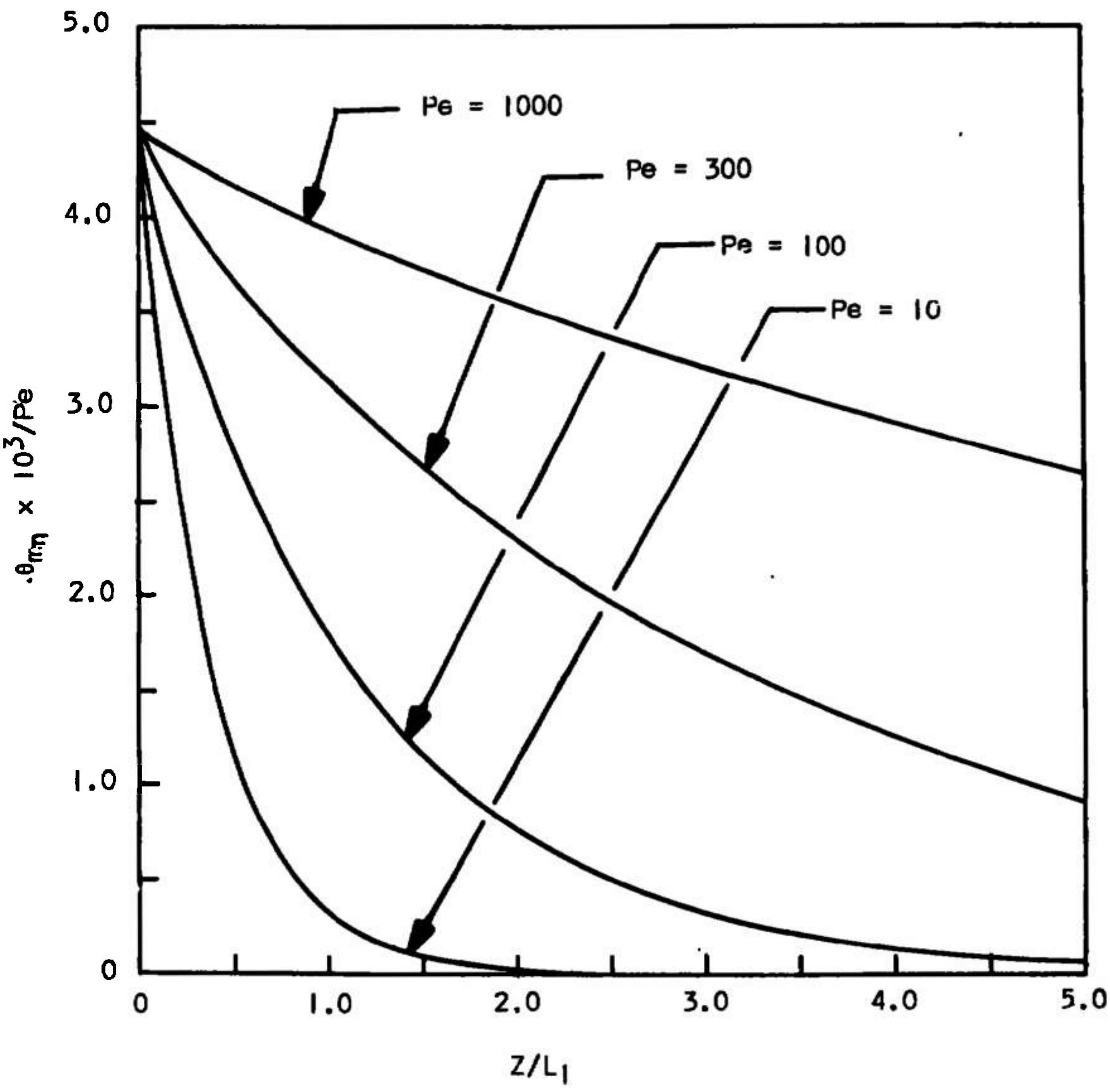

Fig. 17 Bulk Mean Viscous Temperature for Aspect Ratio of 10 and Various Peclet Numbers

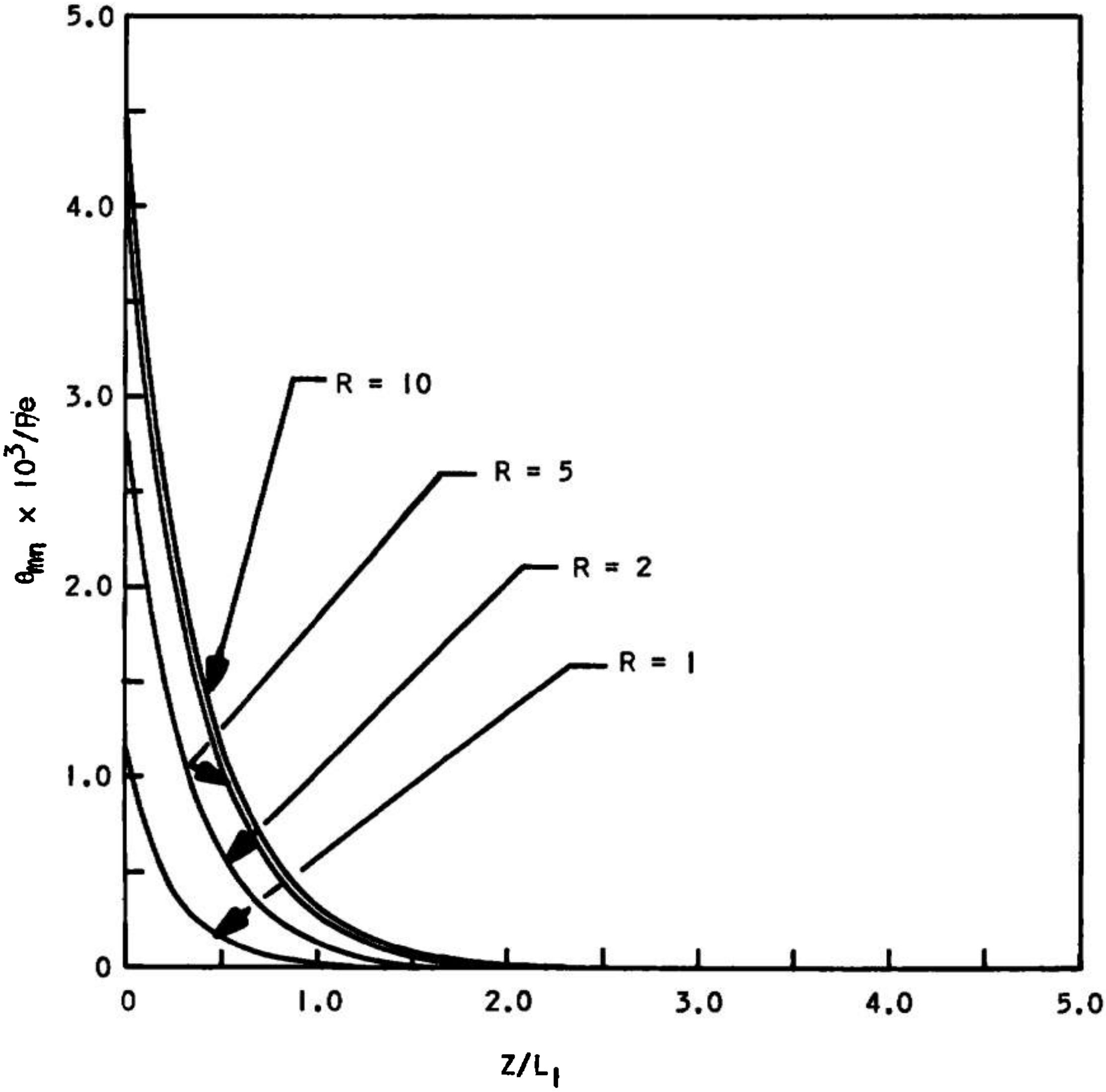

Fig. 18 Bulk Mean Viscous Temperature for Pe of 10 and Various Aspect Ratios

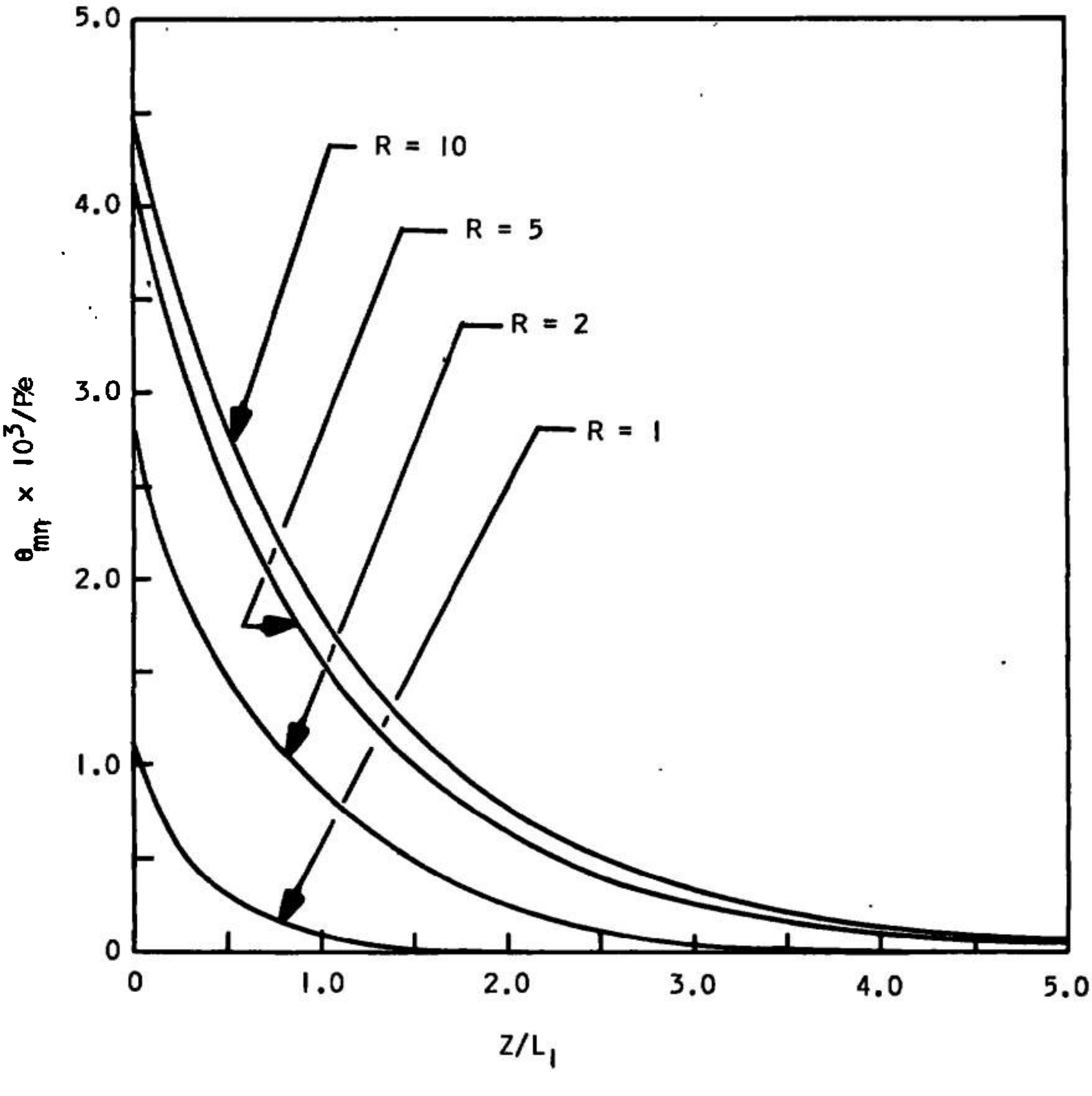

Fig. 19 Bulk Mean Viscous Temperature for Pe of 100 and Various Aspect Ratios

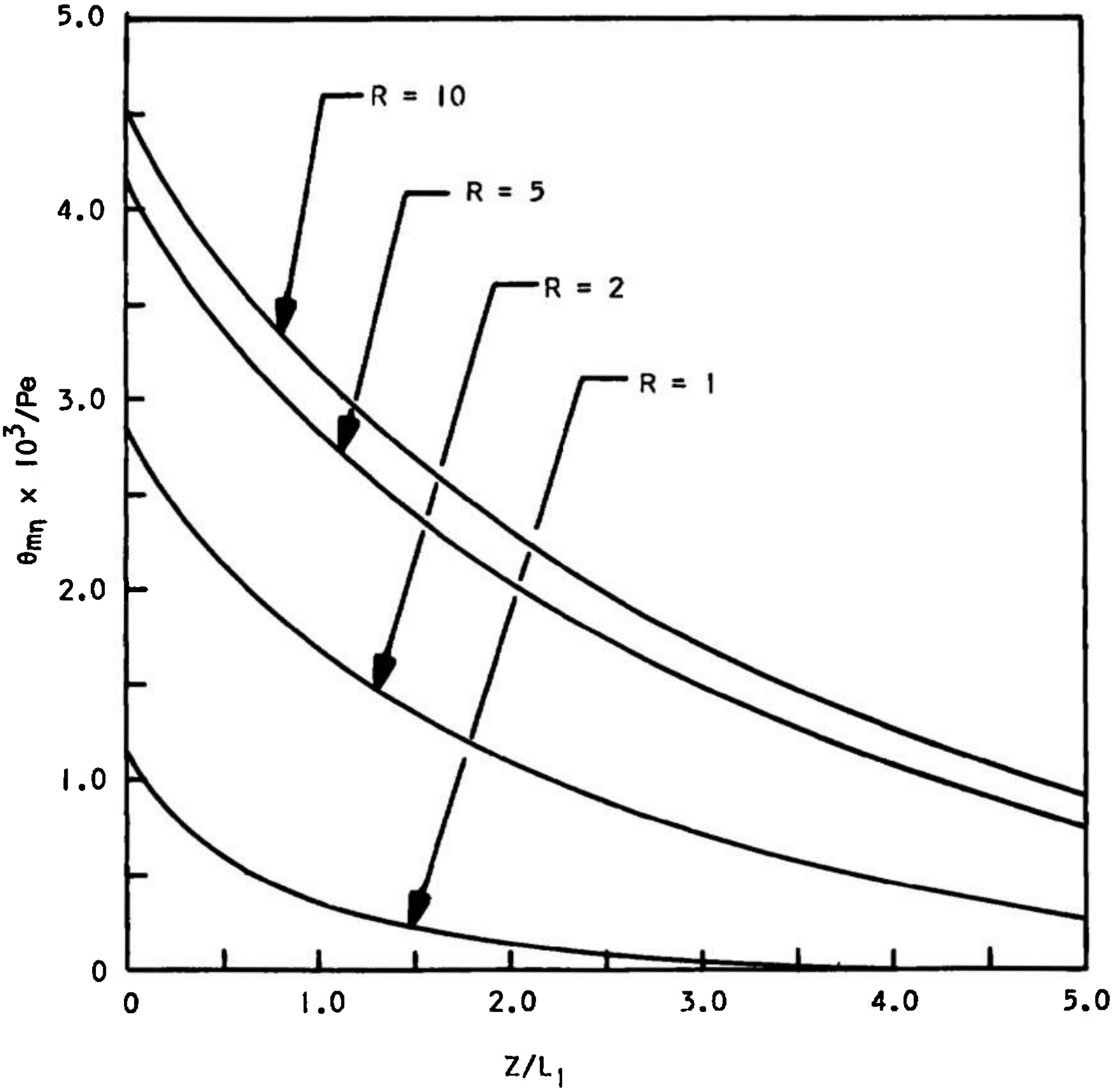

Fig. 20 Bulk Mean Viscous Temperature for Pe of 300 and Various Aspect Ratios

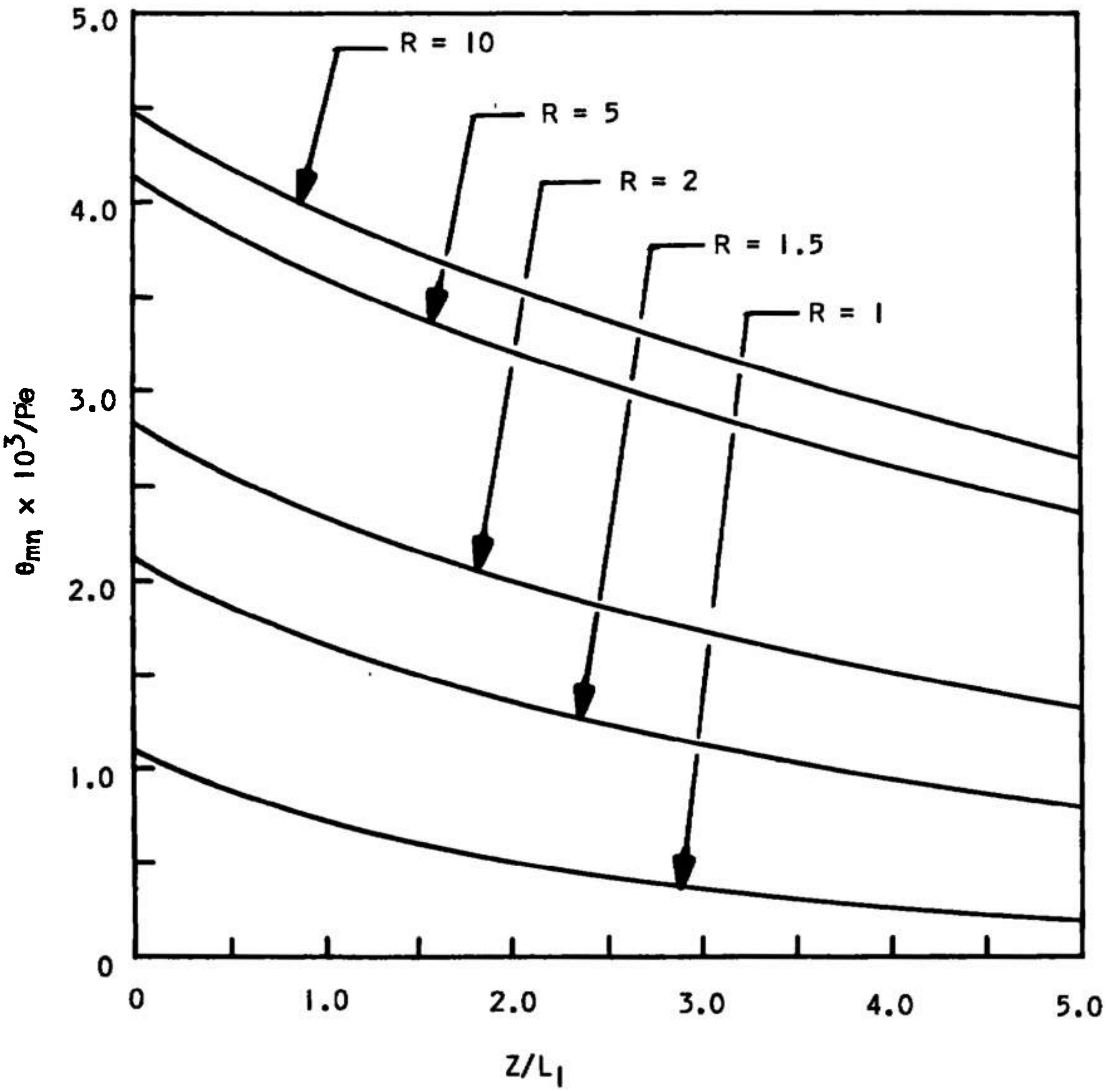

Fig. 21 Bulk Mean Viscous Temperature for Pe of 1000 and Various Aspect Ratios

Figure 22 gives the limiting Nusselt number as a function of aspect ratio for the various Peclet numbers, which is identical for both the viscous and nonviscous components. Plotted in the figure are data from Refs. 13 and 15. The values from Ref. 13 are 2.89 and 2.54 for the aspect ratios of 1 and 2, respectively, and from Ref. 15 are 2.98 and 2.54. These values compare well with the values obtained in this study for Peclet number of 1000, which are 2.981 and 2.548.

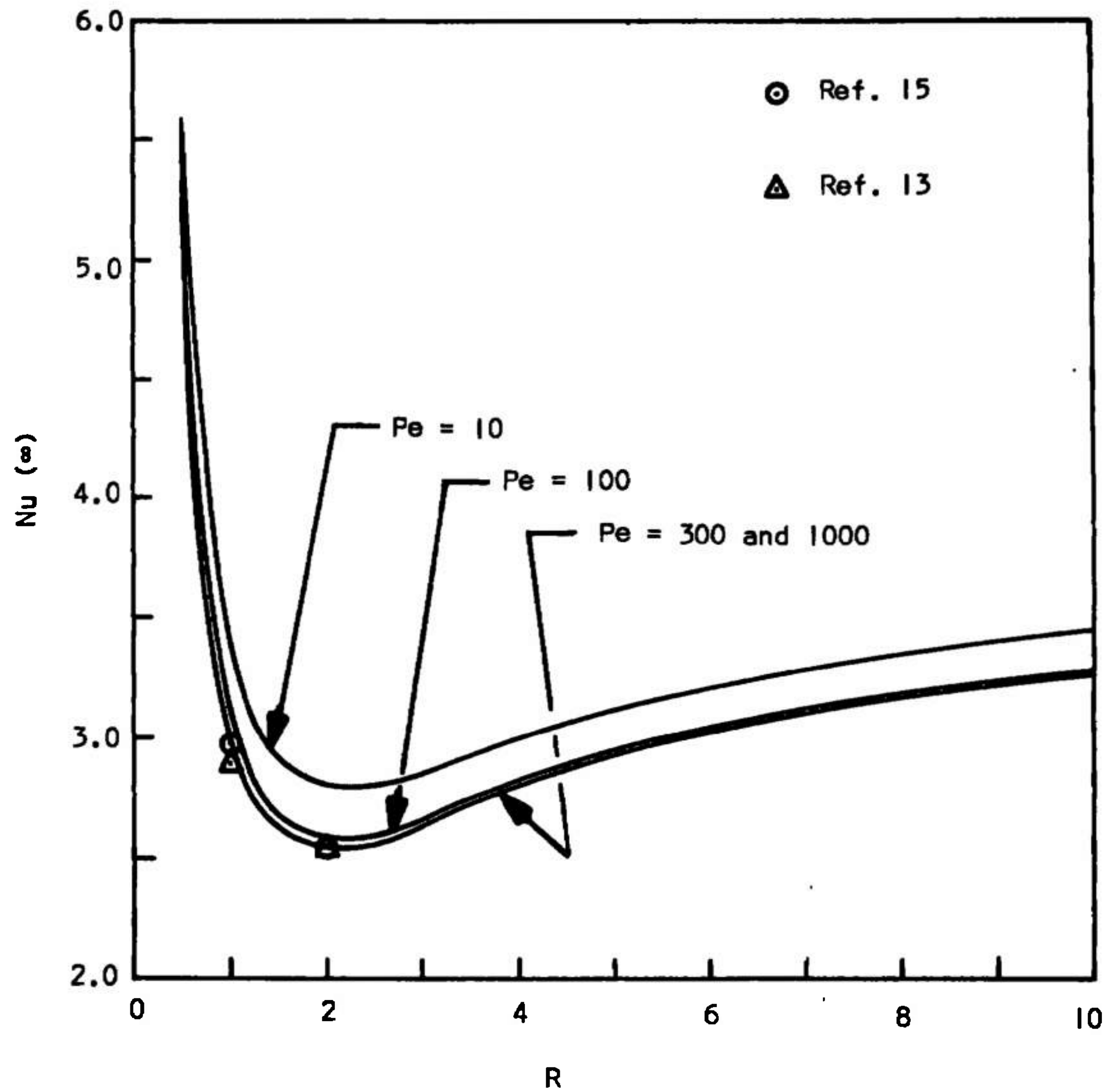

Fig. 22 Limiting Nusselt Number versus Aspect Ratio for Various Peclet Numbers

Figures 23 through 26 give the Nusselt number for the nonviscous component as a function of distance from the duct entrance for aspect ratios of 1, 2, 5, and 10, respectively, with the Peclet number as a parameter.

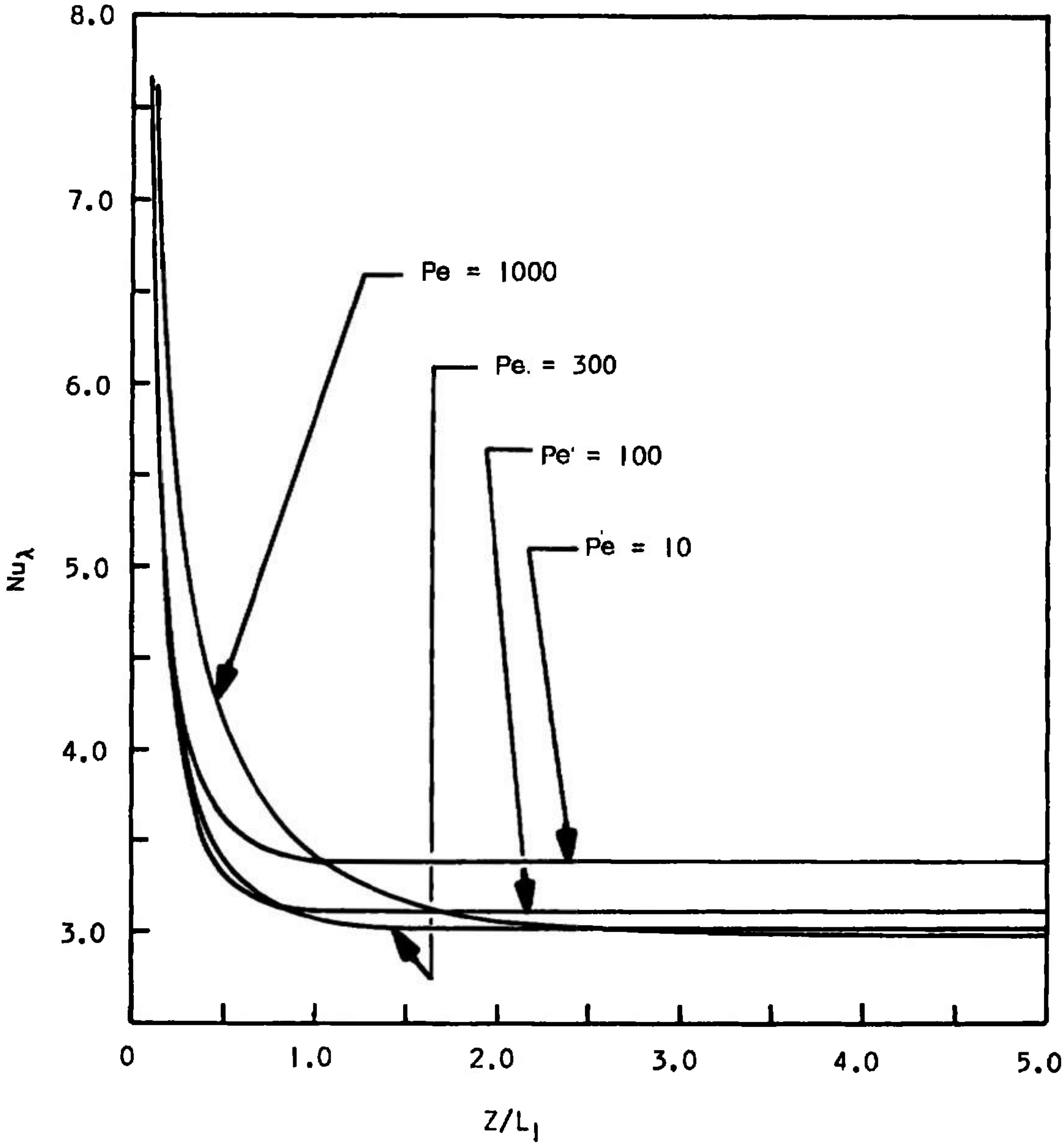

Fig. 23 Nusselt Number for Aspect Ratio of 1 and Various Peclet Numbers

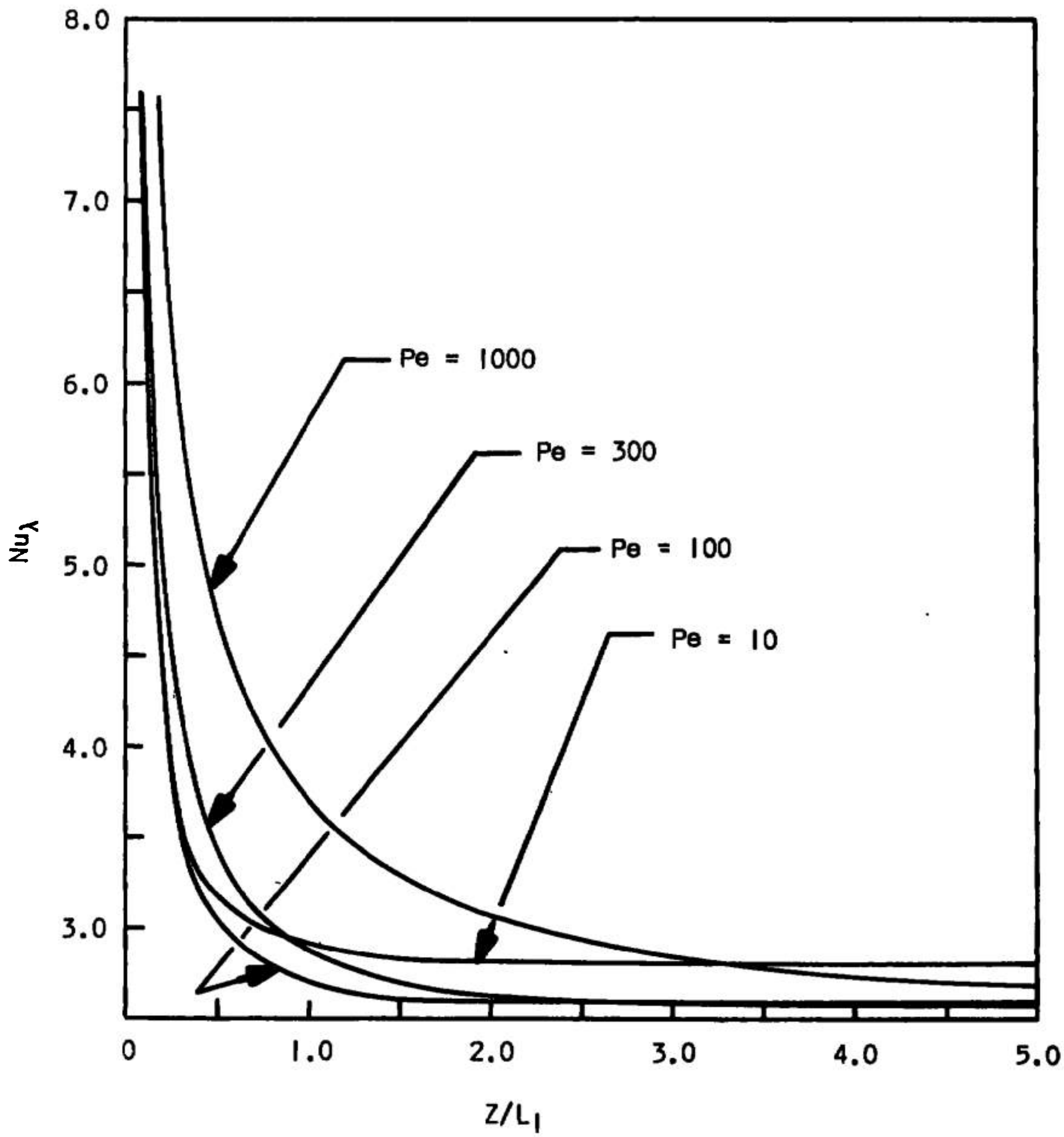

Fig. 24 Nusselt Number for Aspect Ratio of 2 and Various Peclet Numbers

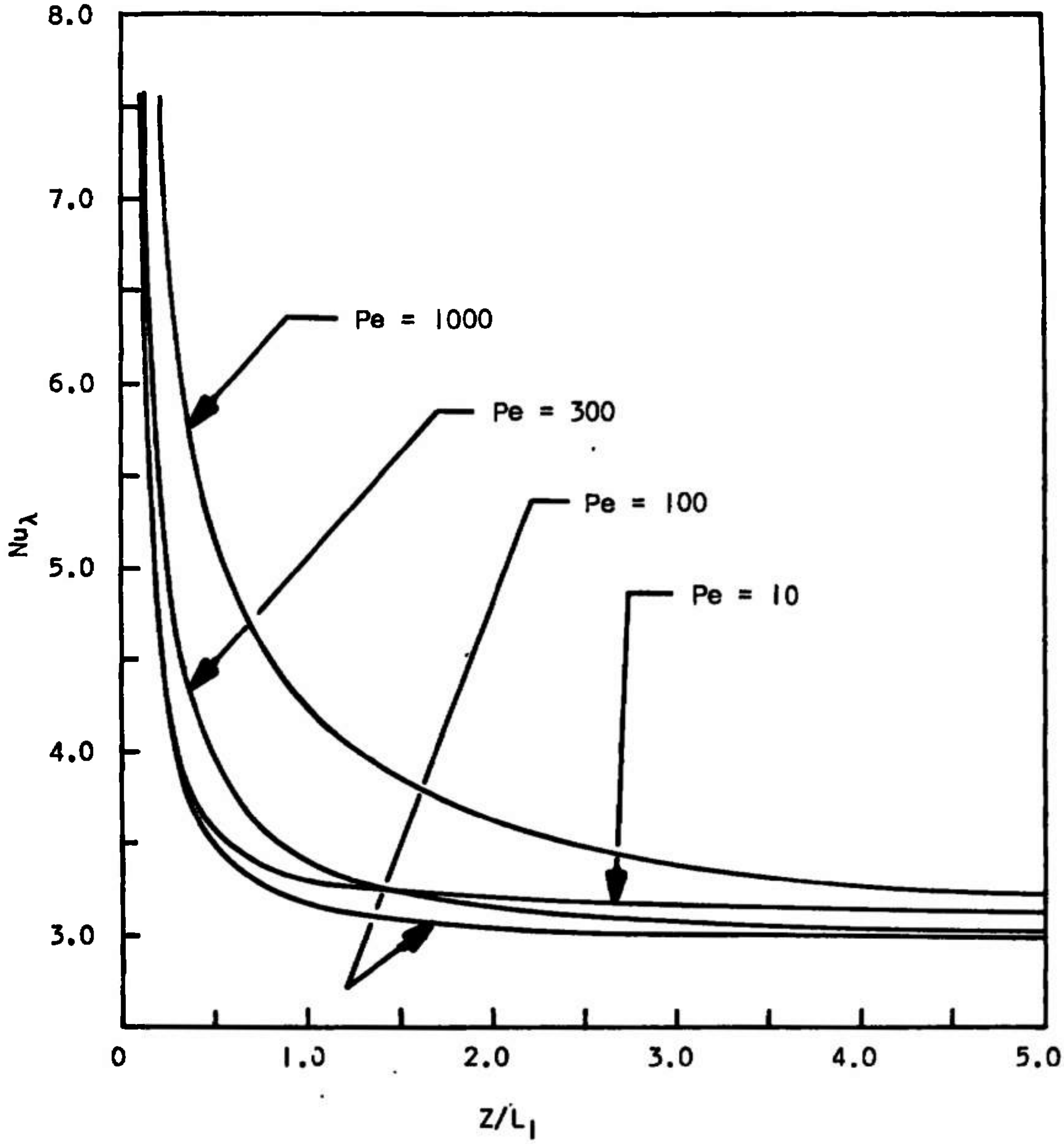

Fig. 25 Nusselt Number for Aspect Ratio of 5 and Various Peclet Numbers

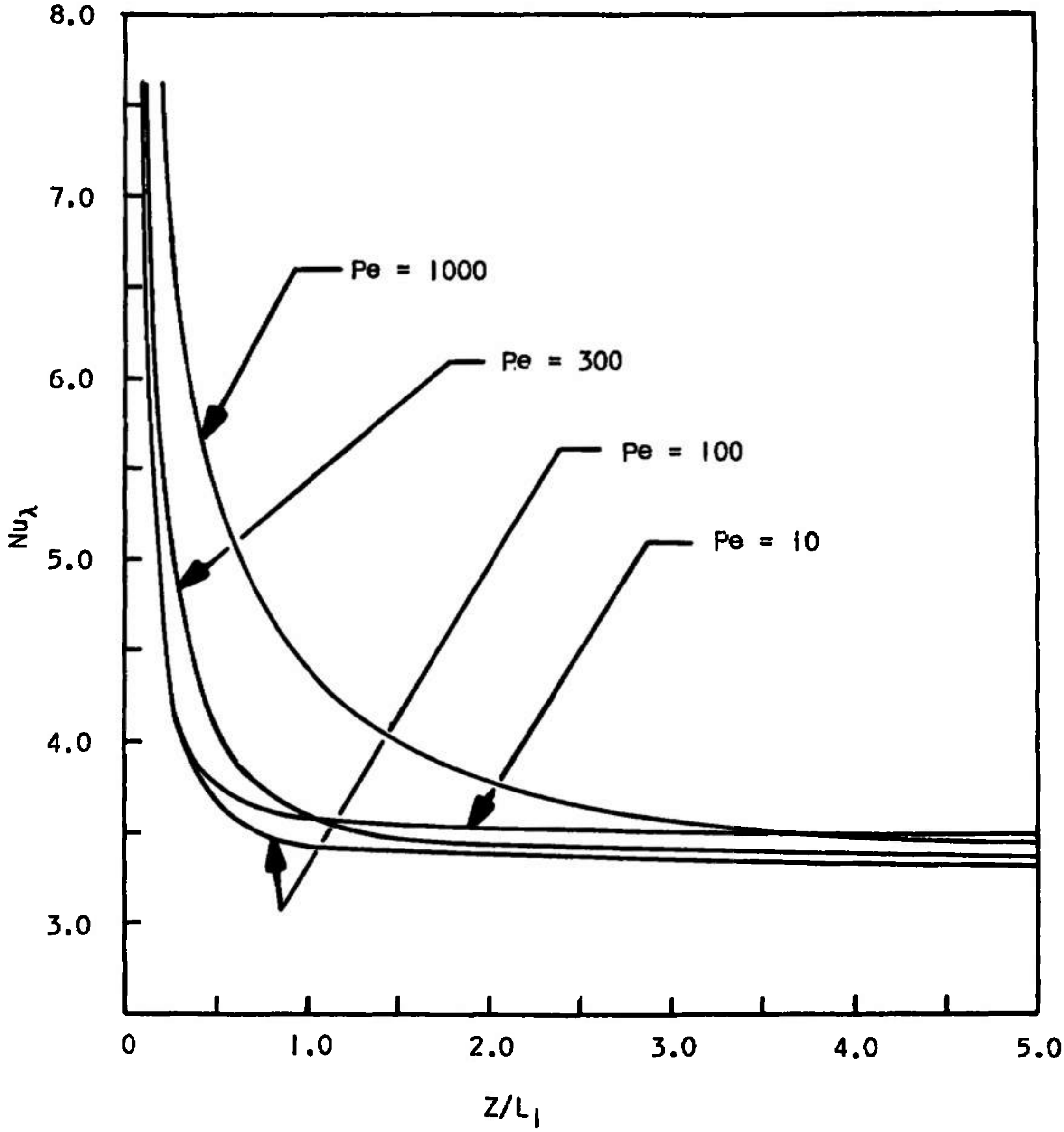

Fig. 26 Nusselt Number for Aspect Ratio of 10 and Various Peclet Numbers

Eraslan and Snyder (Ref. 18) have shown that for viscous dissipation the mean value of the dissipation function over the duct cross section is equal to the product of the pressure gradient and the mean velocity. With the values of bulk mean temperature, mean velocity, and Nusselt number for fully developed viscous flow between parallel plates obtainable from a closed form solution, it is possible to obtain a more accurate estimation of the fully developed viscous dissipation Nusselt number for various aspect ratios by means of the ratio of energy dissipated than is obtainable by differentiating the finite series for the fully developed viscous temperature profile. The Nusselt number may then be shown to be

$$Nu_\eta = Nu_{\eta f} \frac{u_m \; \theta_{m\eta f}}{\theta_{m\eta} \; u_{mf}} \left(\frac{R}{R+T} \right) \tag{34}$$

where the subscript f denotes parallel plate values and where u_m and $\theta_{m\eta}$ may be obtained from Figs. 2 and 13. Figure 27 gives the Nusselt number obtained by this approach as a function of aspect ratio.

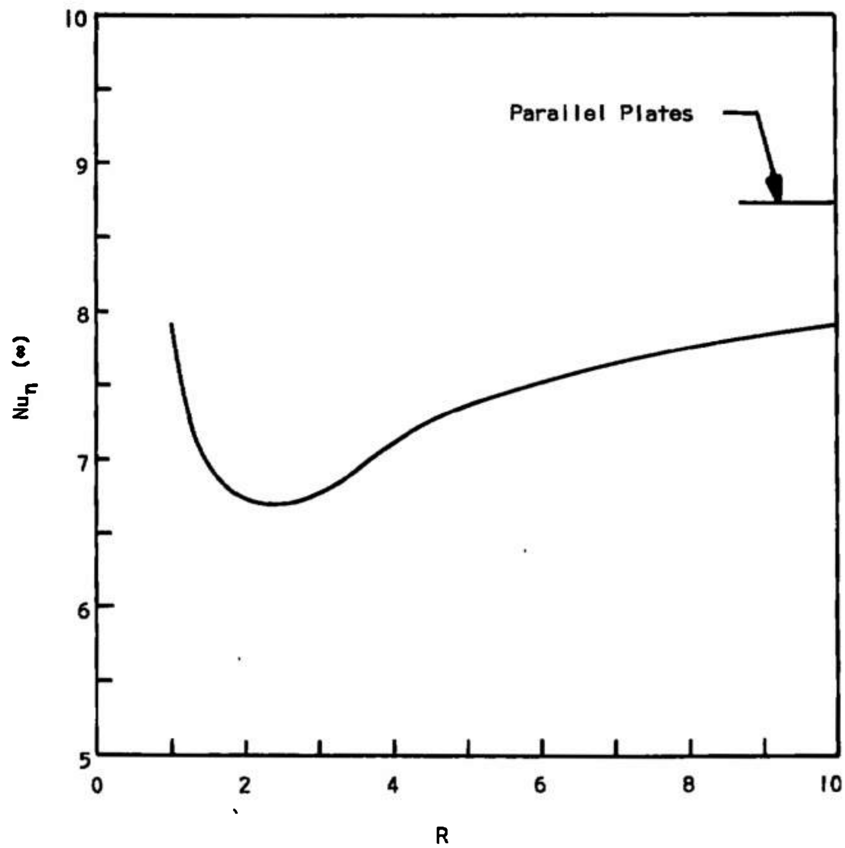

Fig. 27 Nusselt Number versus Aspect Ratio for Fully Developed Viscous Temperature

Figures 28 through 31 give the Nusselt number for the viscous component as a function of distance from the duct entrance for aspect ratios of 1, 2, 5, and 10, respectively, with the Peclet number as a parameter. These figures show that the Nusselt number for the viscous component is not infinite at the duct entrance, as in the case for the nonviscous component, but is given by Fig. 27. Therefore, the curves in Figs. 28 through 31 were started at the duct entrance using the values given by Fig. 27.

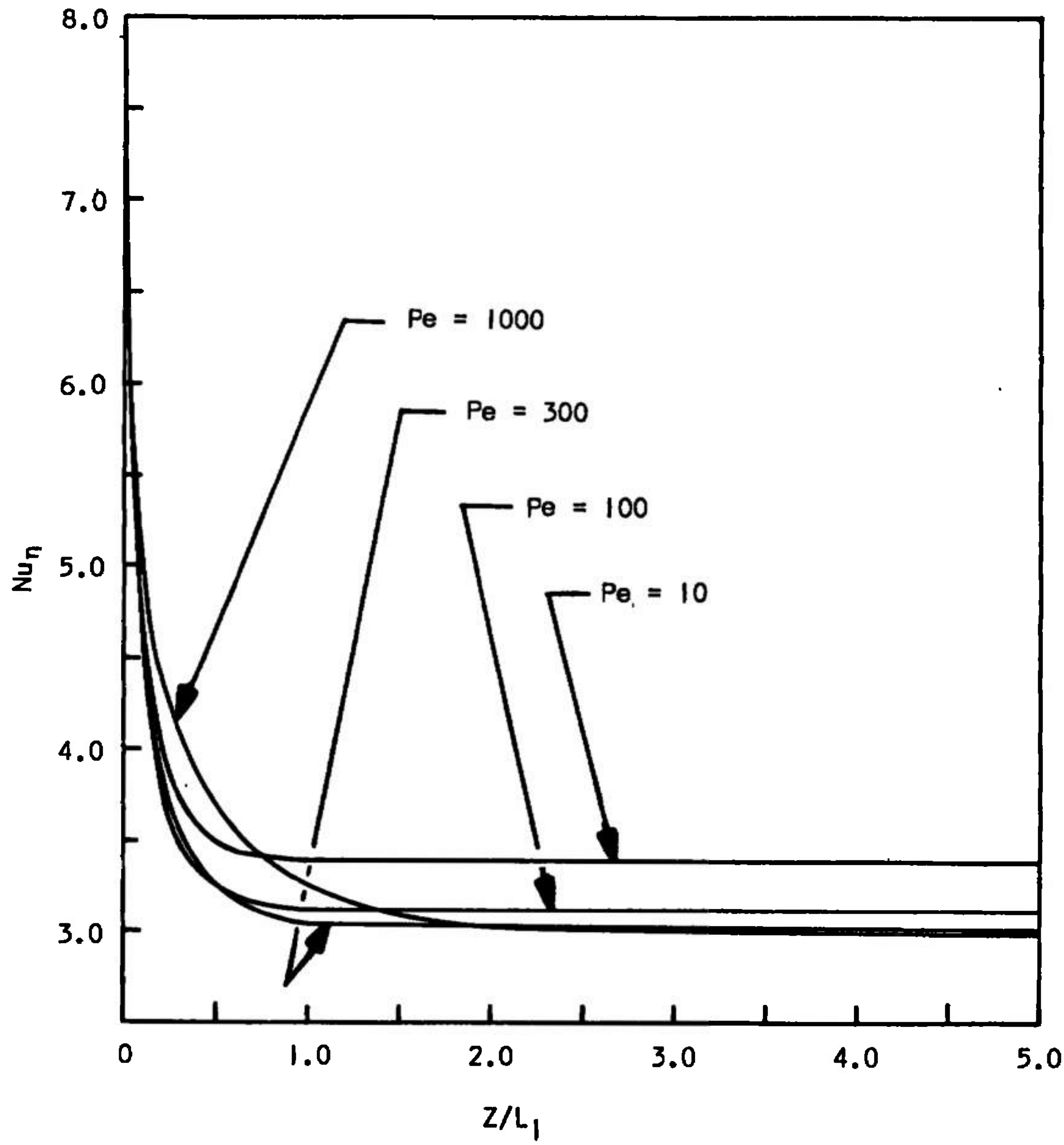

Fig. 28 Viscous Nusselt Number for Aspect Ratio of 1 and Various Peclet Numbers

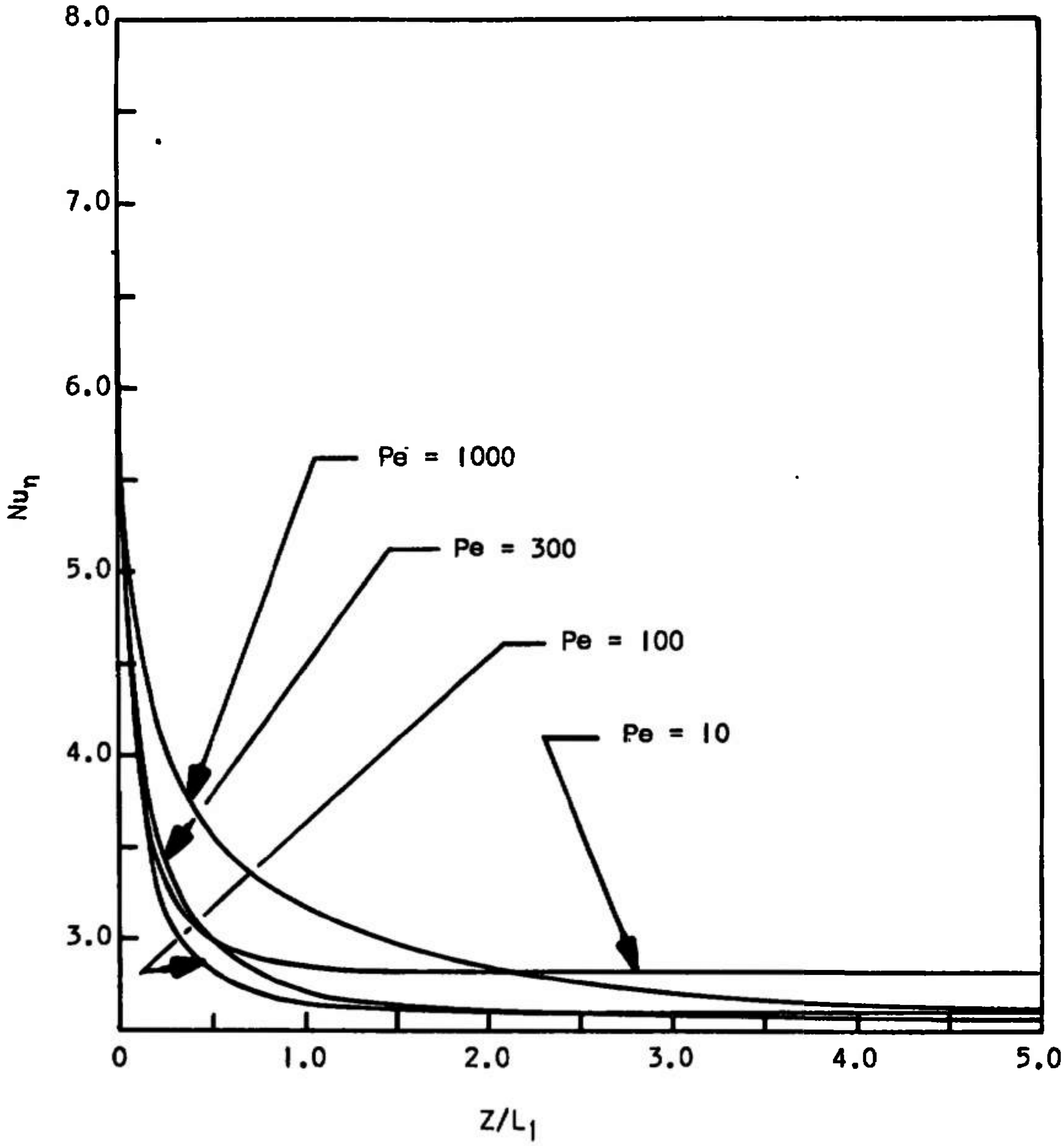

Fig. 29 Viscous Nusselt Number for Aspect Ratio of 2 and Various Peclet Numbers

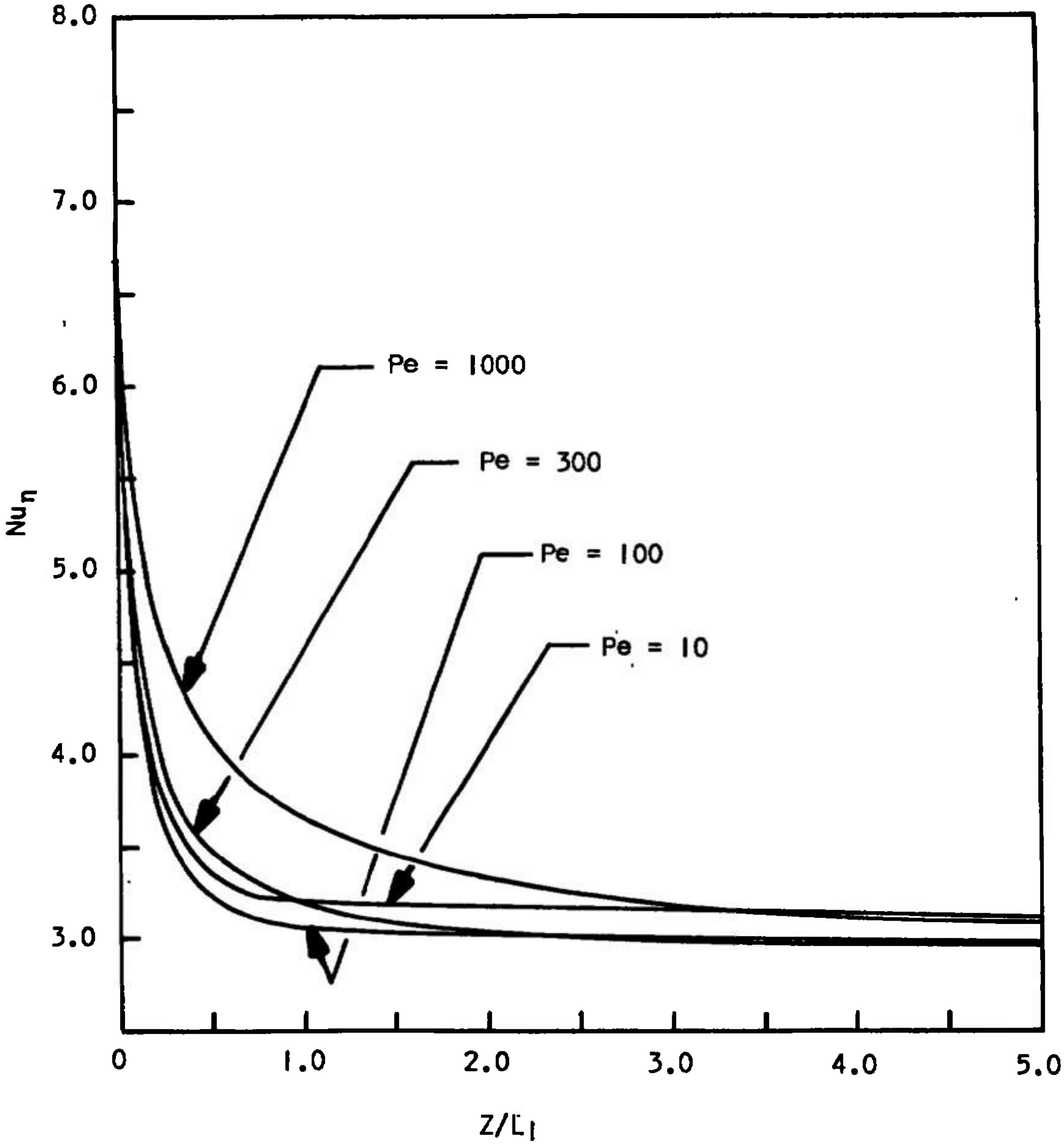

Fig. 30 Viscous Nusselt Number for Aspect Ratio of 5 and Various Peclet Numbers

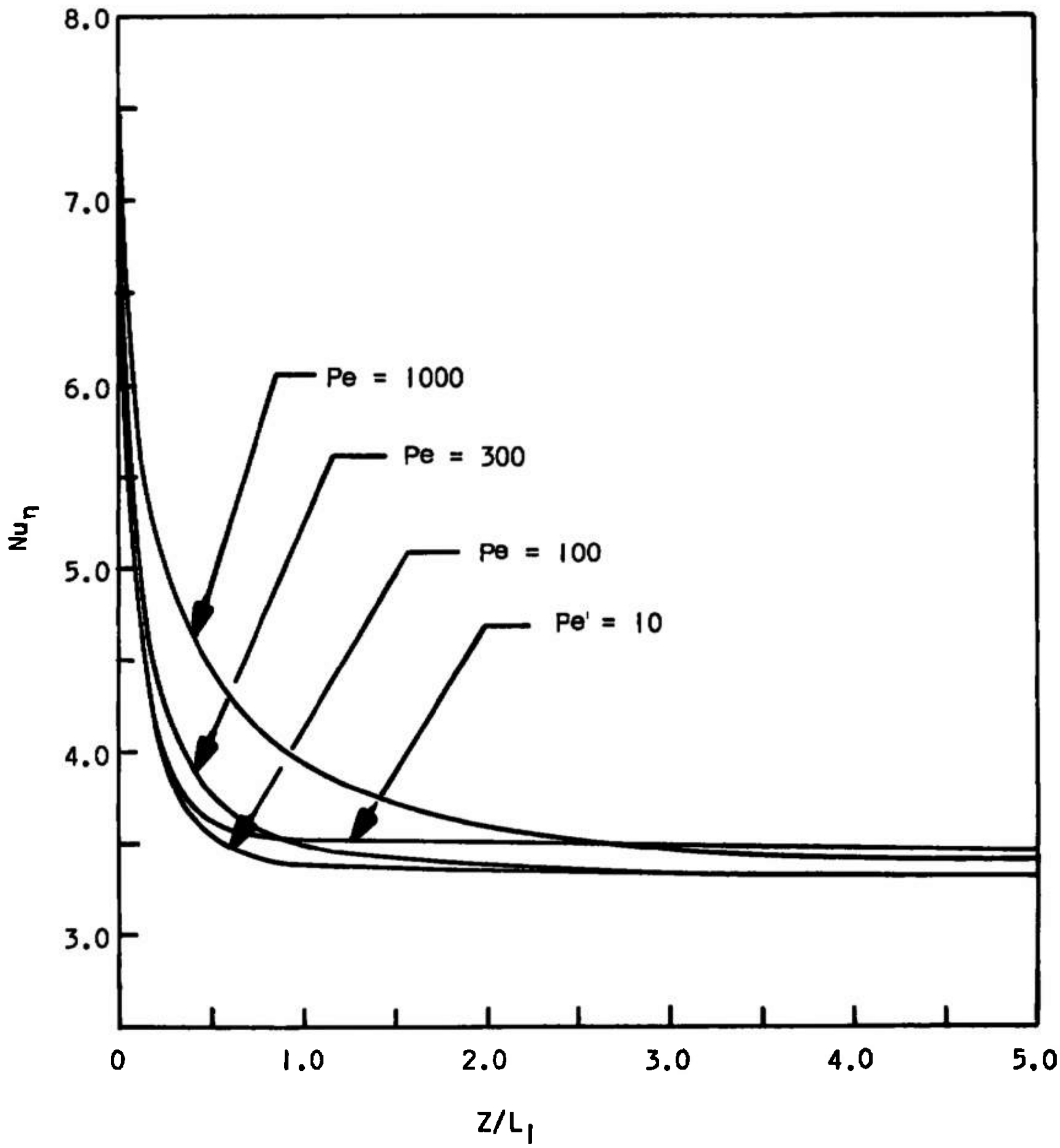

Fig. 31 Viscous Nusselt Number for Apsect Ratio of 10 and Various Peclet Numbers

Figures 32 through 34 give the ratio of heat-transfer rate long-to-short wall as a function of distance from the duct entrance for the nonviscous component and for aspect ratios of 2, 5, and 10, respectively, for the various Peclet numbers. Near the entrance to the duct the solution converges more rapidly to the long than to the short wall. For this region the heat-transfer rate ratio has been estimated by extending the curves as dashed lines.

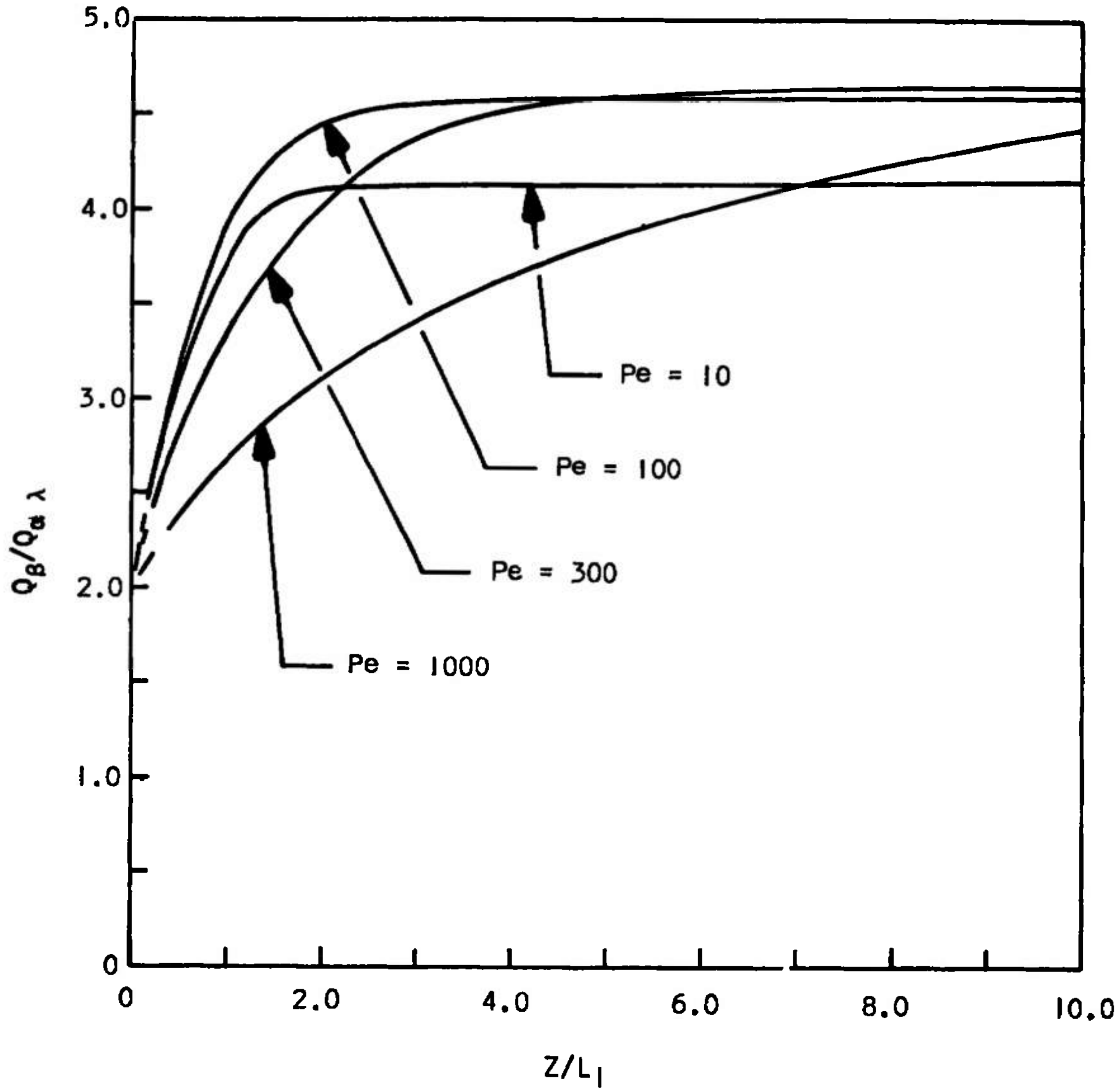

Fig. 32 Heat-Transfer Ratio Long-to-Short Wall for Aspect Ratio of 2 and Various Peclet Numbers

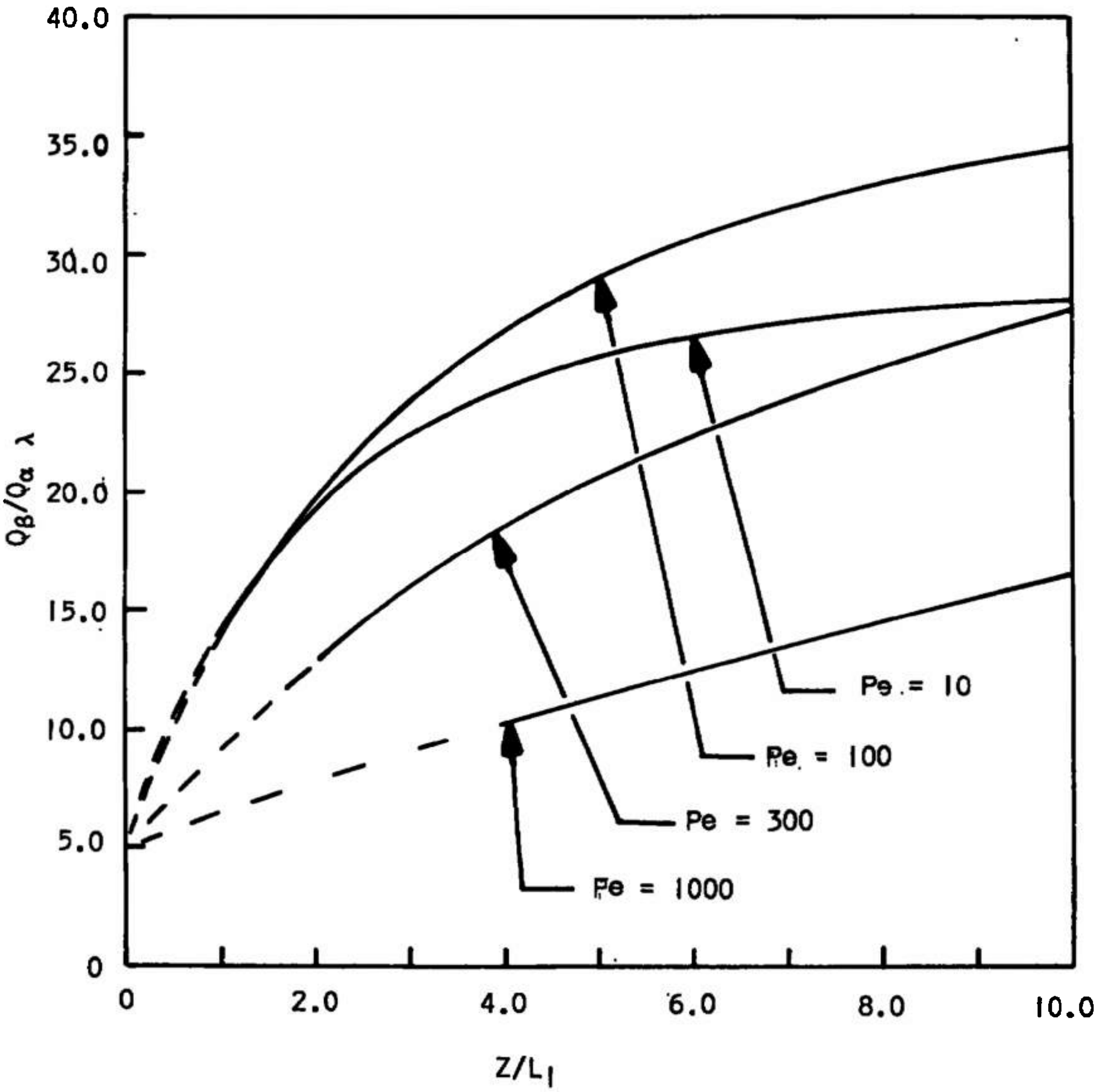

Fig. 33 Heat-Transfer Ratio Long-to-Short Wall for Aspect Ratio of 5 and Various Peclet Numbers

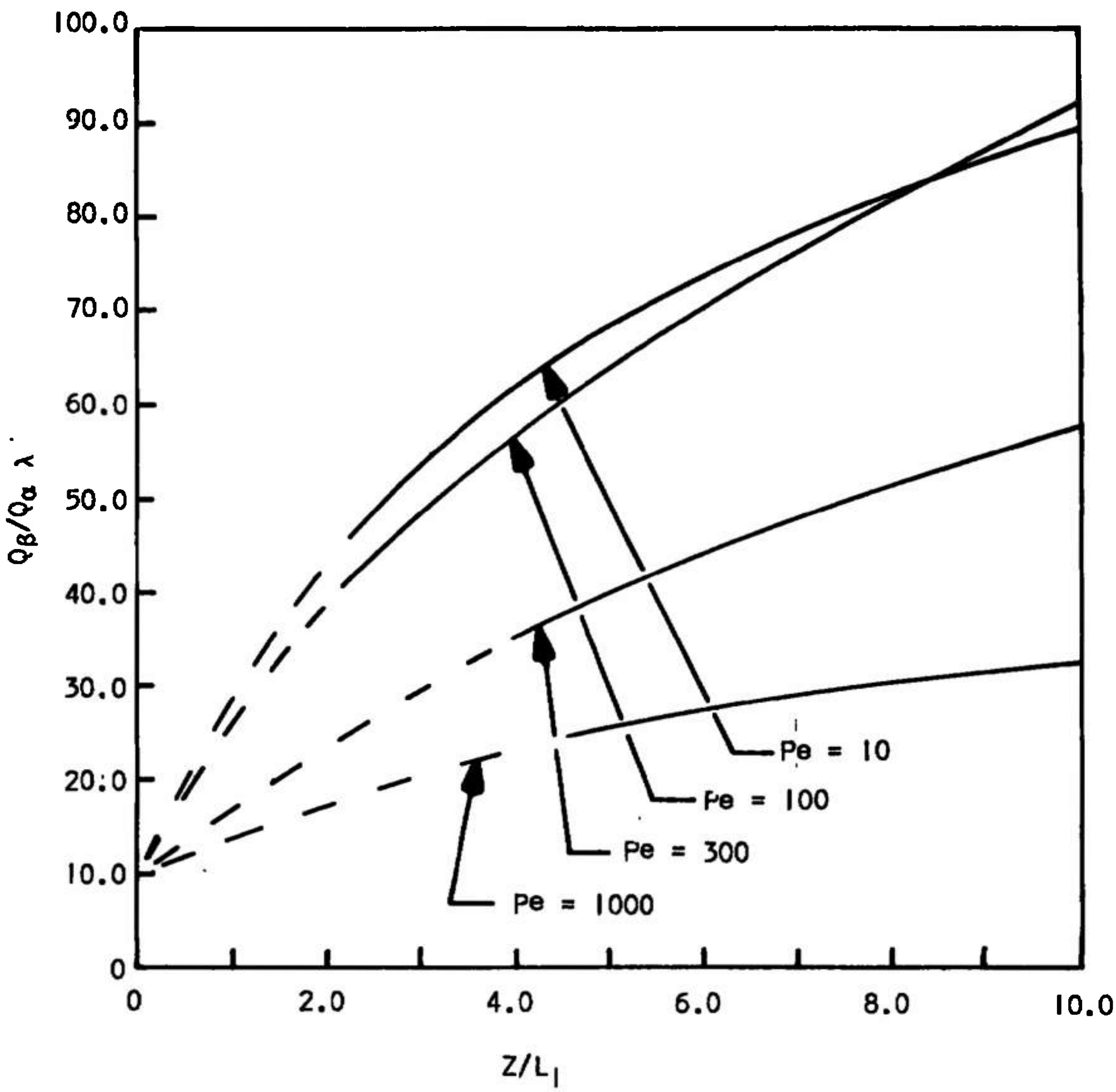

Fig. 34 Heat-Transfer Ratio Long-to-Short Wall for Aspect Ratio of 10 and Various Peclet Numbers

Figures 35 through 37 give the ratio of heat-transfer rate long-to-short wall as a function of distance from the duct entrance for the viscous component and for aspect ratios of 2, 5, and 10, respectively, for the various Peclet numbers. Again, near the entrance of the duct, the ratios have been estimated by dashed lines.

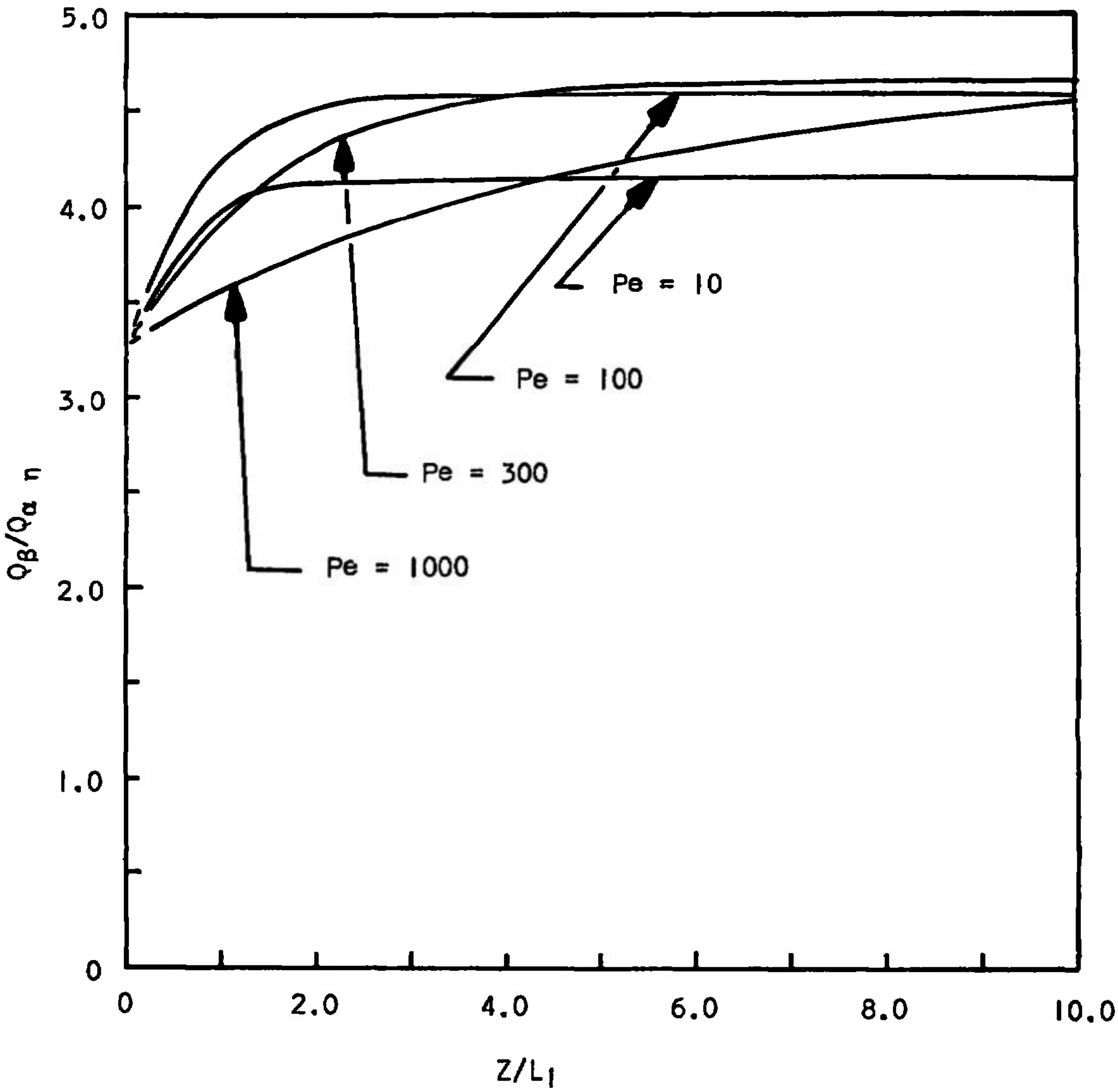

Fig. 35 Viscous Heat-Transfer Ratio Long-to-Short Wall for Aspect Ratio of 2 and Various Peclet Numbers

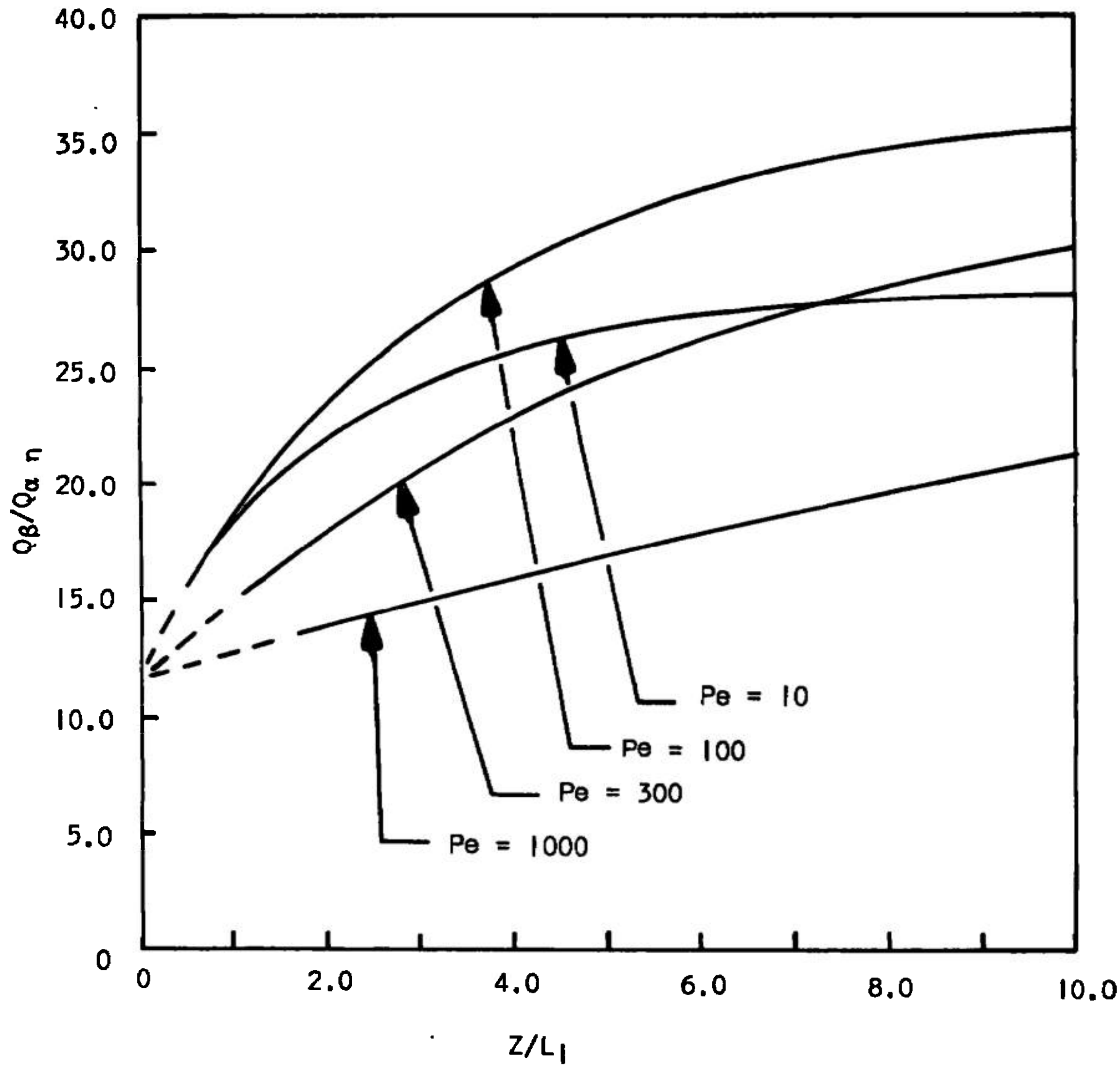

Fig. 36 Viscous Heat-Transfer Ratio Long-to-Short Wall for Aspect Ratio of 5 and Various Peclet Numbers

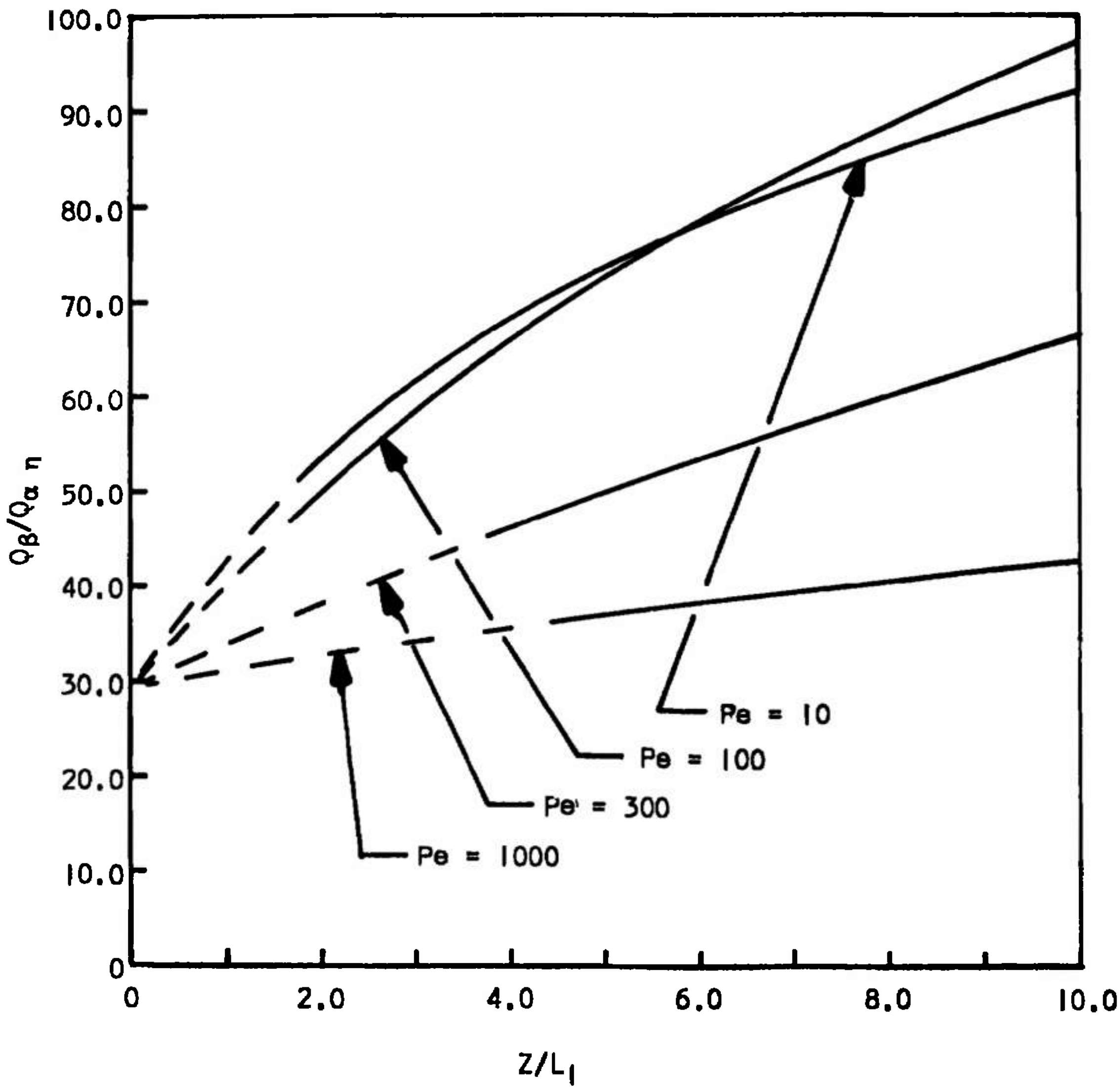

Fig. 37 Viscous Heat-Transfer Ratio Long-to-Short Wall for Aspect Ratio of 10 and Various Peclet Numbers

SECTION V
GENERAL CONCLUSIONS

It is concluded that the B. G. Galerkin method may be used to solve the thermal entrance region heat-transfer problem for laminar flow of a constant property fluid in rectangular ducts with axial conduction and viscous dissipation included if consideration is given to avoiding near multiple roots for the eigenvalues.

The separation of the inlet boundary conditions into viscous dissipation and nonviscous dissipation components allowed a solution to be obtained which was general for the duct entrance to wall temperature difference, $\Delta\theta_T$. The combination of the viscous and nonviscous components gives the solution for the inlet boundary conditions as stated in Section 1.2. The nonviscous component alone will give the solution for no viscous dissipation and uniform temperature inlet, as well as for fully developed viscous temperature flow with a step change in wall temperature in the Z-direction. The viscous component alone will give the solution of viscous temperature flow for the condition of uniform inlet temperature equal to a uniform wall temperature.

The solutions for the thermal entrance region in rectangular ducts with constant wall temperature conditions verify the fact that for low Peclet numbers, the few previous cases solved for no axial conduction can be in considerable error.

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APPENDIX I
SOLUTION OF THE EQUATION OF MOTION FOR THE VELOCITY PROFILE

Given the equation of motion (Eq. (1))

$$\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} = \frac{1}{\mu}\frac{dP}{dZ} \tag{I-1}$$

and nondimensionalizing with

$$W_{ref} = -\frac{L_1{}^2}{\mu}\frac{dP}{dZ} = \frac{W}{u} \tag{I-2}$$

and

$$L_1 = \frac{X}{\alpha} = \frac{Y}{\beta} \tag{I-3}$$

the equation becomes

$$\frac{W_{ref}}{L_1{}^2}\left(\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2}\right) = \frac{1}{\mu}\frac{dP}{dZ} \tag{I-4}$$

or

$$\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} = -1 \tag{I-5}$$

The boundary conditions are

$$u(\Gamma) = 0 \tag{I-6}$$

The solution may be obtained using the finite Fourier sine transform as given by Sneddon (Ref. 19). Taking the transform of both sides of the equation

$$\int_0^1 \int_0^R \left(\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} \right) \sin m \, \pi \alpha \sin \frac{n\pi\beta}{R} \quad d\alpha \, d\beta =$$

$$- \int_0^1 \int_0^R \sin m \, \pi\alpha \, \sin \frac{n\pi\beta}{R} \, d\alpha \, d\beta \tag{I-7}$$

the transform of the left side of the equation with the specified boundary conditions becomes

$$-\pi^2 \left(m^2 + \frac{n^2}{R^2} \right) \frac{R}{4} \; A(m,n) \tag{I-8}$$

The transform of the right side becomes

$$- \frac{R}{mn\pi^2} \left[\cos m\pi - 1 \right] \left[\cos n\pi - 1 \right] \tag{I-9}$$

$$= - \frac{R}{mn\pi^2} \, [2] \, [2] \text{ for } m \text{ and } n \text{ odd} \tag{I-10}$$

$$= 0 \text{ for } m \text{ or } n \text{ even} \tag{I-11}$$

Then

$$A(m,n) = \frac{16}{\pi^4} \left[\frac{R^2}{R^2 m^3 n + mn^3} \right] \tag{I-12}$$

The solution then becomes

$$u = \sum_{m=1,3,5\ldots} \; \sum_{n=1,3,5\ldots} A(m,n) \sin m \, \pi\alpha \, \sin \frac{n\pi\beta}{R} \tag{I-13}$$

or

$$u = \sum_{m=1,3,5\ldots} \sum_{n=1,3,5\ldots} \frac{16}{\pi^4}\left[\frac{R^2}{R^2m^3n+mn^3}\right] \sin m\pi\alpha \sin\frac{n\pi\beta}{R} \qquad \text{(I-14)}$$

The mean velocity may then be determined as

$$u_m = \int_0^1 \int_0^R u \, d\alpha d\beta / R \qquad \text{(I-15)}$$

or

$$u_m = \frac{\displaystyle\int_0^1 \int_0^R \frac{16}{\pi^4}\left[\frac{R^2}{R^2m^3n+mn^3}\right] \sin m\pi\alpha \sin\frac{n\pi\beta}{R} \, d\alpha d\beta}{R} \qquad \text{(I-16)}$$

Then

$$u_m = \frac{64}{\pi^6} \sum_{m=1,3,5\ldots} \sum_{n=1,3,5\ldots} \left[\frac{R^2}{R^2m^4n^2+m^2n^4}\right] \qquad \text{(I-17)}$$

APPENDIX II
SOLUTION OF THE VISCOUS DISSIPATION TEMPERATURE PROFILE

Equation (13) for the viscous dissipation temperature profile

$$\frac{\partial^2 \theta_n}{\partial \alpha^2} + \frac{\partial^2 \theta_n}{\partial \beta^2} = - Pe \left[\left(\frac{\partial u}{\partial \alpha} \right)^2 + \left(\frac{\partial u}{\partial \beta} \right)^2 \right] \qquad \text{(II-1)}$$

with boundary conditions

$$\theta_n(\Gamma, \xi) = 0$$

may be solved using the finite Fourier sine transform. Taking the transform of both sides as

$$\int_0^1 \int_0^R \left(\frac{\partial^2 \theta_n}{\partial \alpha^2} + \frac{\partial^2 \theta_n}{\partial \beta^2} \right) \sin p\, \pi\alpha \sin q \frac{\pi\beta}{R}\, d\alpha d\beta =$$

$$- Pe \int_0^1 \int_0^R \left[\left(\frac{\partial u}{\partial \alpha} \right)^2 + \left(\frac{\partial u}{\partial \beta} \right)^2 \right] \sin p\, \pi\alpha \sin q \frac{\pi\beta}{R}\, d\alpha d\beta \qquad \text{(II-2)}$$

the transform of the left side as given by Sneddon (Ref. 19) becomes

$$- \pi^2 \left(p^2 + \frac{q^2}{R^2} \right) \frac{R}{4}\, \overline{f}(p,q) \qquad \text{(II-3)}$$

From the velocity solution

$$\left(\frac{\partial u}{\partial \alpha} \right)^2 = \left(\frac{16}{\pi^4} \right) \sum_{m=1,3,5} \sum_{n=1,3,5} \sum_{s=1,3,5} \sum_{t=1,3,5} \left[\frac{\pi R^2}{R^2 m^2 n + n^3} \right] \left[\frac{\pi R^2}{R^2 s^2 t + t^3} \right]$$

$$\cos m\, \pi\alpha \sin n \frac{\pi\beta}{R} \cos s\, \pi\alpha \sin t \frac{\pi\beta}{R} \qquad \text{(II-4)}$$

and

$$\left(\frac{\partial u}{\partial \beta}\right)^2 + \left(\frac{16}{\pi^4}\right) \sum_{m=1,3,5} \sum_{n=1,3,5} \sum_{s=1,3,5} \sum_{t=1,3,5} \left[\frac{\pi R}{R^2m^3+mn^2}\right] \left[\frac{\pi R}{R^2s^3+st^2}\right]$$

$$\sin m \pi\alpha \cos n \frac{\pi\beta}{R} \sin s \pi\alpha \cos t \frac{\pi\beta}{R} \qquad \text{(II-5)}$$

The right side then becomes

$$-Re \left(\frac{16}{4}\right)^2 \left[\sum_m \sum_n \sum_s \sum_t \left(\frac{\pi^2 R^4}{(R^2m^2n+n^3)(R^2s^2t+t^3)}\right)\right.$$

$$\int_0^1 \int_0^R \cos m \pi\alpha \cos s \pi\alpha \sin p \pi\alpha \sin n \frac{\pi\beta}{R} \sin t \frac{\pi\beta}{R} \sin q \frac{\pi\beta}{R}$$

$$d\alpha d\beta + \sum_m \sum_n \sum_s \sum_t \left(\frac{\pi^2 R^2}{(R^2m^3+mn^2)(R^2s^3+st^2)}\right)$$

$$\left.\int_0^1 \int_0^R \sin m \pi\alpha \sin s \pi\alpha \sin p \pi\alpha \cos n \frac{\pi\beta}{R} \cos t \frac{\pi\beta}{R} \sin q \frac{\pi\beta}{R} d\alpha d\beta\right] \qquad \text{(II-6)}$$

By use of trigonometric identities for multiple angles the first integral becomes

$$\int_0^1 \int_0^R \frac{1}{16} \left[\sin (m + (s+p))\pi\alpha - \sin (m - (s+p))\pi\alpha\right.$$

$$\left.- \sin (m + (s-p))\pi\alpha + \sin (m - (s-p))\pi\alpha\right]\left[\sin (n + (t-q))\frac{\pi\beta}{R}\right.$$

$$\left.+ \sin (n-(t-q))\frac{\pi\beta}{R} - \sin (n+(t+q))\frac{\pi\beta}{R} - \sin (n-(t+q))\frac{\pi\beta}{R}\right]$$

$$d\alpha\, d\beta \qquad \text{(II-7)}$$

which, on integration, becomes

$$\frac{R}{4\pi^2}\left[\frac{1}{m+(s+p)} - \frac{1}{m-(s+p)} - \frac{1}{m+(s-p)} + \frac{1}{m-(s-p)}\right]$$

$$\left[\frac{1}{m+(t-q)} + \frac{1}{n-(t-q)} - \frac{1}{n+(t+q)} - \frac{1}{n-(t+q)}\right] \qquad \text{(II-8)}$$

and exists only for p and q = 1, 3, 5 ---- since m, n, s, and t are odd and

$$\int_0^1 \sin n\,\pi\alpha d\alpha \equiv 0 \text{ for } n \text{ even}$$

By the same process the second integral becomes

$$\frac{R}{4\pi^2}\left[\frac{1}{m+(s-p)} + \frac{1}{m-(s-p)} - \frac{1}{m+(s+p)} - \frac{1}{m-(s+p)}\right]$$

$$\left[\frac{1}{n+(t+q)} - \frac{1}{n-(t+q)} - \frac{1}{n+(t-q)} + \frac{1}{n-(t-q)}\right] \qquad \text{(II-9)}$$

and exists only for p and q = 1, 3, 5 ------- .

The final solution then becomes

$$\theta_n(\alpha,\beta) = Pe \sum_{p=1,3,5..} \sum_{q=1,3,5..} \bar{f}_s(p,q)$$

$$\sin p\,\pi\alpha \sin q\,\frac{\pi\beta}{R} \qquad \text{(II-10)}$$

where

$$\bar{f}_s(p,q) = \frac{256\ R^4}{\pi^{10}} \left(\frac{1}{R^2p^2+q^2}\right) \left[\sum_m \sum_n \sum_s \sum_t \right.$$

$$\left(\frac{R^2}{(R^2m^2n+n^3)(R^2s^2t+t^3)}\right)\left[\frac{1}{m+(s+p)}-\frac{1}{m-(s+p)}-\frac{1}{m+(s-p)}+\frac{1}{m-(s-p)}\right]$$

$$\left[\frac{1}{n+(t-q)}+\frac{1}{n-(t-q)}-\frac{1}{n+(t+q)}-\frac{1}{n-(t+q)}\right]+\left(\frac{1}{(R^2m^3+mn^2)(R^2s^3+st^2)}\right)$$

$$\left[\frac{1}{m+(s-p)}+\frac{1}{m-(s-p)}-\frac{1}{m+(s+p)}-\frac{1}{m-(s+p)}\right]\left[\frac{1}{n+(t+q)}-\frac{1}{n-(t+q)}-\frac{1}{n+(t-q)}\right.$$

$$\left.\left.-\frac{1}{n-(t-q)}\right]\right] \qquad \text{(II-11)}$$

APPENDIX III
SOLUTION OF THE INTEGRALS FOR I_1, I_2, AND I_3

For approximating functions of the form

$$\phi_i = \sin p\, \pi\alpha \sin q \frac{\pi\beta}{R} \tag{III-1}$$

where p and q are functions of i and for the velocity solution

$$u = \frac{16}{\pi^4} \sum_{m=1,3,5\ldots} \sum_{n=1,3,5\ldots} \left(\frac{R^2}{R^2 m^3 n + m n^3} \right) \sin m\,\pi\alpha \sin n \frac{\pi\beta}{R} \tag{III-2}$$

and

$$I_1(k,i) = \int_0^1 \int_0^R \nabla^2 \phi_i(\alpha,\beta)\, \phi_k(\alpha,\beta)\, d\alpha d\beta \tag{III-3}$$

$$\nabla^2 \phi_i(\alpha,\beta) = -\pi^2 (p^2 + q^2/R^2) \sin p\,\pi\alpha \sin q \frac{\pi\beta}{R} \tag{III-4}$$

then

$$I_1(k,i) = \int_0^1 \int_0^R -\pi^2 (p^2 + q^2/R^2) \sin p\,\pi\alpha \sin s\,\pi\alpha \sin q \frac{\pi\beta}{R}$$

$$\sin t \frac{\pi\beta}{R}\, d\alpha d\beta \tag{III-5}$$

where p = s and q = t for k = i

$$I_1(k,i) = -\pi^2 (p^2 + q^2/R^2) \frac{R}{4} \quad \text{for } k = i \tag{III-6}$$

$$I_1(k,i) = 0 \text{ for } k \neq i \tag{III-7}$$

For

$$I_2(k,i) = \int_0^1 \int_0^R u(\alpha,\beta)\ \phi_i(\alpha,\beta)\ \phi_k(\alpha,\beta)\ d\alpha\ d\beta \qquad \text{(III-8)}$$

then

$$I_2(k,i) = \frac{16}{\pi^4} \int_0^1 \int_0^R \sum_{m=1,3,5} \ \sum_{n=1,3,5} \left(\frac{R^2}{R^2 m^3 n + m n^3}\right)$$

$$\sin m\pi\alpha \ \sin p\pi\alpha \ \sin s\pi\alpha \ \sin n\frac{\pi\beta}{R} \ \sin q\frac{\pi\beta}{R} \ \sin t\ \frac{\pi\beta}{R} d\alpha\ d\beta \qquad \text{(III-9)}$$

which, with the use of trigonometric identities, becomes

$$I_2(k,i) = \frac{16}{\pi^4} \int_0^1 \int_0^R \sum_{m=1,3,5..} \ \sum_{n=1,3,5..} \frac{1}{16} \left(\frac{R^2}{R^2 m^3 n + m n^3}\right)$$

$$\left[\sin (m+(p-s))\pi\alpha + \sin(m-(p-s))\pi\alpha - \sin(m+(p+s))\pi\alpha - \sin(m-(p+s))\pi\alpha\right]$$

$$\left[\sin (n+(q-t))\frac{\pi\beta}{R} + \sin (n-(q-t))\frac{\pi\beta}{R} - \sin (n+(q+t))\frac{\pi\beta}{R}\right.$$

$$\left. - \sin (n-(q+t))\frac{\pi\beta}{R}\right] d\alpha\ d\beta \qquad \text{(III-10)}$$

Since p, q, s, and t may be specified as odd integers because of symmetry about the duct centerlines, the solution then becomes

$$I_2(k,i) = \frac{4}{\pi^6} \sum_{m=1,3,5..} \ \sum_{n=1,3,5..} \left(\frac{R^3}{R^2 m^3 n + m n^3}\right)\left(\frac{1}{m+(p-s)}\right.$$

$$\left. + \frac{1}{m-(p-s)} - \frac{1}{m+(p+s)} - \frac{1}{m-(p+s)}\right)\left(\frac{1}{n+(q-t)} + \frac{1}{n-(q-t)} - \frac{1}{n+(q+t)}\right.$$

$$- \frac{1}{n-(q+t)} \Bigg) \tag{III-11}$$

where i corresponds to the pair of integers (p, q), and k corresponds to the pair of integers (s, t). For

$$I_3(k,i) = \int_0^1 \int_0^R \phi_i(\alpha,\beta)\ \phi_k(\alpha,\beta)\ d\alpha\ d\beta \tag{III-12}$$

then

$$I_3(k,i) = \int_0^1 \int_0^R \sin p\ \pi\alpha\ \sin s\ \pi\alpha\ \sin q\ \frac{\pi\beta}{R}\ \sin t\ \frac{\pi\beta}{R}\ d\alpha\ d\beta \tag{III-13}$$

$$I_3(k,i) = R/4 \text{ for } k = i \tag{III-14}$$

$$= 0 \text{ for } k \neq i \tag{III-15}$$

For the special case of the square duct where the eigenfunctions used are

$$\phi_i(\alpha,\beta) = \sin p\ \pi\alpha\ \sin q\ \pi\beta + \sin q\ \pi\alpha\ \sin p\ \pi\beta \tag{III-16}$$

the integrals may be shown to be

$$I_1(k,i) = -\pi^2(p^2 + q^2)/2 \quad k = i \text{ and } p \neq q \tag{III-17}$$

$$I_1(k,i) = -\pi^2(p^2 + q^2) \quad k = i \text{ and } p = q \tag{III-18}$$

$$I_1(k,i) = 0 \quad k \neq i \tag{III-19}$$

$$
\begin{aligned}
I_2(k,i) = \frac{4}{\pi^6} \sum_{m=1,3,5..} \sum_{n=1,3,5..} \left(\frac{1}{m^3 n + m n^3}\right)
&\left[\left(\frac{1}{m+(p-s)} + \frac{1}{m-(p-s)} - \frac{1}{m+(p+s)} - \frac{1}{m-(p+s)}\right)\left(\frac{1}{n+(q-t)}\right.\right. \\
&+ \left.\frac{1}{n-(q-t)} - \frac{1}{n+(q+t)} - \frac{1}{n-(q+t)}\right) + \left(\frac{1}{m+(q-s)} + \frac{1}{m-(q-s)}\right. \\
&- \left.\frac{1}{m+(q+s)} - \frac{1}{m-(q+s)}\right)\left(\frac{1}{n+(p-t)} + \frac{1}{n-(p-t)} - \frac{1}{n+(p+t)}\right. \\
&- \left.\frac{1}{n-(p+t)}\right) + \left(\frac{1}{m+(p-t)} + \frac{1}{m-(p-t)} - \frac{1}{m+(p+t)} - \frac{1}{m-(p+t)}\right) \\
&\left(\frac{1}{n+(q-s)} + \frac{1}{n-(q-s)} - \frac{1}{n+(q+s)} - \frac{1}{n-(q+s)}\right) + \left(\frac{1}{m+(q-t)}\right. \\
&+ \left.\frac{1}{m-(q-t)} - \frac{1}{m+(q+t)} - \frac{1}{m-(q+t)}\right)\left(\frac{1}{n+(p-s)} + \frac{1}{n-(p-s)}\right. \\
&- \left.\left.\frac{1}{n+(p+s)} - \frac{1}{n-(p+s)}\right)\right]
\end{aligned}
\qquad \text{(III-20)}
$$

where i corresponds to the pair of integers (p, q), and k corresponds to the pair of integers (s, t).

$$I_3(k,i) = 1 \qquad k = i \text{ and } p = q \qquad \text{(III-21)}$$

$$I_3(k,i) = 1/2 \qquad k = i \text{ and } p \neq q \qquad \text{(III-22)}$$

$$I_3(k,i) = 0 \qquad k \neq i \qquad \text{(III-23)}$$

APPENDIX IV
MISCELLANEOUS INTEGRALS

For the integrals given in this appendix the functions, $\phi_i(\alpha, \beta)$, are those given by Eqs. (26) and (27).

The integral $I(\ell, m)$ as given by Eq. (24),

$$I(\ell,m) = \int_0^1 \int_0^R \left[\phi_1(\alpha,\beta) + \sum_{i=2}^{N} C_i^{(m)} \phi_i(\alpha,\beta) \right]$$

$$\left[\phi_1(\alpha,\beta) + \sum_{j=2}^{N} C_j^{(\ell)} \phi_j(\alpha,\beta) \right] d\alpha \, d\beta \qquad \text{(IV-1)}$$

for the nonsquare cases becomes, on integration,

$$I(\ell,m) = \frac{R}{4} \left[1 + \sum_{i=2}^{N} C_i^{(m)} C_i^{(\ell)} \right] \qquad \text{(IV-2)}$$

and for the square cases becomes

$$I(\ell,m) = \left[1 + \sum_{i=2}^{N} C^* \right] \qquad \text{(IV-3)}$$

where

$$C^* = C_i^{(m)} C_i^{(\ell)} \text{ for } p = q$$

$$= \frac{1}{2} C_i^{(m)} C_i^{(\ell)} \text{ for } p \neq q$$

The integral for $J_\lambda(\ell)$ as given by Eq. (30),

$$J_\lambda(\ell) = \int_0^1 \int_0^R \left[\phi_1(\alpha,\beta) + \sum_{j=2}^{N} C_j^{(\ell)} \phi_j(\alpha,\beta) \right] d\alpha \, d\beta \qquad \text{(IV-4)}$$

for the nonsquare cases becomes, on integration,

$$J_\lambda(\ell) = \frac{4R}{\pi^2} \left[1 + \sum_{j=2}^{N} C_j^{(\ell)}/pq \right] \qquad \text{(IV-5)}$$

and for the square cases becomes

$$J_\lambda(\ell) = \frac{8}{\pi^2} \left[1 + \sum_{j=2}^{N} C_j^{(\ell)}/pq \right] \qquad \text{(IV-6)}$$

The integral for $J_\eta(\ell)$ as given by Eq. (31),

$$J_\eta(\ell) = \int_0^1 \int_0^R \sum_{s=1,3,5..} \sum_{t=1,3,5..} \overline{f}_s(s,t) \sin s\pi\alpha \sin t\frac{\pi\beta}{R}$$

$$\left[\phi_1(\alpha,\beta) + \sum_{j=2}^{N} C_j^{(\ell)} \phi_j(\alpha,\beta) \right] d\alpha \, d\beta \qquad \text{(IV-7)}$$

for the nonsquare cases becomes, on integration,

$$J_\eta(\ell) = \frac{R}{4} \left[\overline{f}_s(1,1) + \sum_{j=2}^{N} \overline{f}_s(p,q)\, C_j^{(\ell)} \right] \qquad \text{(IV-8)}$$

and for the square cases becomes

$$J_\eta(\ell) = \left[2\,\overline{f}_s(1,1) + 2\sum_{j=2}^{N} \overline{f}_s(p,q)\,C_j^{(\ell)}\right] \qquad \text{(IV-9)}$$

The bulk mean temperature which is defined as

$$\theta_m(\xi) = \frac{\int_0^1\int_0^R \theta(\alpha,\beta,\xi)\,u(\alpha,\beta)\,d\alpha d\beta}{\int_0^1\int_0^R u(\alpha,\beta)\,d\alpha\,d\beta} \qquad \text{(IV-10)}$$

for the nonsquare cases becomes, on integration, for the nonviscous component

$$\theta_{m\lambda}(\xi) = \frac{4R\Delta\theta_T}{\pi^4}\sum_{n=1}^{N} a_{1\lambda}^{(n)}\left[\frac{R^2}{R^2+1} + \sum_{i=2}^{N} C_i^{(n)}\frac{R^3}{R^2p^3q+pq^3}\right] e^{-B_n\xi} \Big/ u_m \qquad \text{(IV-11)}$$

$$\theta_{m\eta}(\xi) = \frac{4RP_e}{\pi^4}\sum_{n=1}^{N} a_{1n}^{(n)}\left[\frac{R^2}{R^2+1} + \sum_{i=2}^{N} C_i^{(n)}\frac{R^2}{R^2p^3q+pq^3}\right] e^{-B_n\xi} \Big/ u_m \qquad \text{(IV-12)}$$

and for the square cases becomes, for the nonviscous component,

$$\theta_{m\lambda}(\xi) = \frac{16\Delta\theta_T}{\pi^4}\sum_{n=1}^{N} a_{1\lambda}^{(n)}\left[\frac{1}{4} + \frac{1}{2}\sum_{i=2}^{N}\frac{C_i^{(n)}}{p^3q+pq^3}\right] e^{-B_n\xi} \Big/ u_m \qquad \text{(IV-13)}$$

and for the viscous component

$$\theta_{mn}(\xi) = \frac{16\,P_e}{\pi^4}\sum_{n=1}^{N} a_{1n}^{(n)}\left[\frac{1}{4} + \frac{1}{2}\sum_{i=2}^{N}\frac{C_i^{(n)}}{p^3q+pq^3}\right] e^{-B_n\xi} \Big/ u_m \qquad \text{(IV-14)}$$

The ratio of heat-transfer rate per unit length at the duct walls to the thermal conductivity may be written, for the α or short wall, as

$$Q_\alpha/K = \int_0^1 \left.\frac{\partial\theta}{\partial\beta}\right|_{\beta=0} d\alpha \tag{IV-15}$$

and for the β or long wall as

$$Q_\beta/K = \int_0^R \left.\frac{\partial\theta}{\partial\alpha}\right|_{\alpha=0} d\beta \tag{IV-16}$$

and becomes, on differentiation and integration, for the nonsquare cases

$$Q_\alpha/K = \sum_{n=1}^{N} a_1^{(n)} \left[\frac{2}{4} + 2 \sum_{i=2}^{N} C_i^{(n)} \frac{q}{Rp}\right] e^{-B_n\xi} \tag{IV-17}$$

$$Q_\beta/K = \sum_{n=1}^{N} a_1^{(n)} \left[2R + 2 \sum_{i=2}^{N} C_i^{(n)} \frac{pR}{q}\right] e^{-B_n\xi} \tag{IV-18}$$

where the coefficients $a_1^{(n)}$ become $a_{1\lambda}^{(n)}$ and $a_{1\eta}^{(n)}$ for the nonviscous and viscous components, respectively, and for the square cases becomes

$$Q_\alpha/K = Q_\beta/K = \sum_{n=1}^{N} a_1^{(n)} \left[4 + 2 \sum_{i=2}^{N} \left(\frac{p}{q} + \frac{q}{p}\right)\right] e^{-B_n\xi} \tag{IV-19}$$

with the above-mentioned condition on the coefficient $a_1^{(n)}$.

With the evaluation of the heat-transfer rates, the Nusselt numbers may now be written, for the nonviscous component, as

$$Nu_\lambda = \frac{hL_1}{K}$$

$$= \left(\frac{Q_{\alpha\lambda}}{K} + \frac{Q_{\beta\lambda}}{K}\right) \Big/ (R+1)\,\theta_{m\lambda} \tag{IV-20}$$

and for the viscous component as

$$Nu_\eta = \frac{h\,L_1}{K}$$

$$= \left(\frac{Q_{\alpha\eta}}{K} + \frac{Q_{\beta\eta}}{K}\right) \Big/ (R+1)\,\theta_{m\eta} \qquad \text{(IV-21)}$$

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13 ABSTRACT

In this investigation the thermal entrance region problem is solved for flow in rectangular ducts of various aspect ratios and for various Peclet numbers. The assumptions under which the problem is solved are steady, fully developed laminar velocity profile, constant fluid properties of viscosity, density, specific heat and thermal conductivity, constant wall temperature, and a uniform inlet fluid temperature. Included in the solution are axial conduction and viscous dissipation. The method of B. G. Galerkin is used to formulate an approximate series solution of the problem. The data presented include bulk mean temperature, local Nusselt number, and the ratio of the local heat-transfer rate to the long wall to the local heat-transfer rate to the short wall. It is concluded that for low Peclet numbers, neglecting the axial conduction term leads to considerable error in the solution.

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